Geometric Optimal Control of a Class of Quantum Systems

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Minimum Time Control of a Quantum Bit
Optimal Control Problem

Consider a two level quantum system with Hamiltonian $H = u_x \sigma_x + u_y \sigma_y$, where $\sigma_x$ and $\sigma_y$ are the Pauli matrices subject to a control field $u_x, u_y$ bounded in magnitude, i.e., $u_x^2 + u_y^2 \leq 1$. 
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Optimal control problem

For the Schrödinger operator equation

$$\dot{X} = -i u_x \sigma_x X - i u_y \sigma_y X, \quad X(0) = 1,$$

given a final condition $X_f \in SU(2)$, find the time optimal control with $\|u\|^2 \leq 1$ driving the state $X$ from $1$ to $X_f$, in minimum time.
Sub-Riemannian Geometry Interpretation

The time optimal control problem can be interpreted as the problem of finding the length minimizing sub-Riemannian geodesics of $SU(2)$ where the sub-Riemannian structure is specified by a set of vector fields $\Delta := \{-i\sigma_x X, -i\sigma_y X\}$.
Motivations

- In applications to **Quantum Computation** minimum time is a natural requirement.
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- Fast dynamics is a way to counteract the effect of the environment (**decoherence**)
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- Fast dynamics is a way to counteract the effect of the environment (decoherence).
- The trade-off between energy and time characterizes the Quantum Speed Limit and gives the bound in the time-energy uncertainty relations (cf., e.g., M.M. Taddei Thesis (2014) arxiv.org/pdf/1407.4343).
Fact 1

Let $D$ be a diagonal matrix in $SU(2)$ and let $X = X(t)$ be a trajectory (geodesic) $1 \to X_f$ in $SU(2)$, in $[0, T]$, $\|u\| \leq 1$. $DXD^\dagger$ is a trajectory (geodesic) $1 \to DX_fD^\dagger$, in time $[0, T]$, $\|u\| \leq 1$. 
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Reduction to the Unit Disc in the Complex Plane

For $X \in SU(2)$

$$X := \begin{pmatrix} x & y \\ -y^* & x^* \end{pmatrix}, \quad |x|^2 + |y|^2 = 1$$

the action of $D$, $X \to DXD^\dagger$ only affects the off-diagonal elements. The complex number $x$ (an element of the unit disc in the complex plane) determines the equivalence class (orbit).
The application of the necessary conditions of optimality (Pontryagin Maximum Principle) to this problem gives the explicit form of the optimal controls, which are

\[ u_x = \sin(\lambda t) \], \[ u_y = \cos(\lambda t) \].
Fact 2

The application of the necessary conditions of optimality (Pontryagin Maximum Principle) to this problem gives the explicit form of the optimal controls, which are

\begin{align*}
u_x &= \sin(\omega t + \phi), \\
u_y &= -\cos(\omega t + \phi)
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    u_x &= \sin(\omega t + \phi), \\
    u_y &= -\cos(\omega t + \phi)
\end{align*}
\]

The problem is to find the parameters \(\omega\) and \(\phi\) to achieve the desired final condition in minimum time.
The Schrödinger equation can be explicitly integrated to give

$$X(t) := 
\begin{pmatrix}
e^{-i\omega t} \left( \cos(\sqrt{\omega^2 + 1} t) + i \frac{\omega}{\sqrt{\omega^2 + 1}} \sin(\sqrt{\omega^2 + 1} t) \right) & * \\
-e^{i\omega t - 2i\phi} \frac{1}{\sqrt{\omega^2 + 1}} \sin(\sqrt{\omega^2 + 1} t) & *
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The (1,1) element of this matrix represents the equivalence class of the candidate optimal trajectory. Notice that it depends only on \( \omega \). Therefore we only have to tune \( \omega \) to reach the desired orbit.
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The (1,1) element of this matrix represents the equivalence class of the candidate optimal trajectory. Notice that it depends only on \( \omega \). Therefore we only have to tune \( \omega \) to reach the desired orbit. To do this we use the relation between the time optimal control and the reachable sets.
Boundary of reachable sets

For $0 < t \leq \pi$ the parametric curve $x_t = x_t(\omega)$ in the unit complex disc given by the $(1, 1)$ entry of $X = X(t)$ gives the boundary of the reachable set at time $t$

$$x_t(\omega) = e^{-i\omega t} \left( \cos(\sqrt{\omega^2 + 1} t) + i \frac{\omega}{\sqrt{\omega^2 + 1}} \sin(\sqrt{\omega^2 + 1} t) \right)$$
Examples of Boundaries of Reachable Sets for $t = 0.5$, $t = 1$, $t = 2$
Method to find the Optimal Control

1. Step 1: Identify the $(1,1)$ element in the desired final condition $X_f \in SU(2)$ and its position as a point $P_f$ in the unit disc. This is the orbit to be reached.
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2. Step 2: Find the first $t \in (0, \pi]$ so that $P_f \in x_t(\omega)$ and the corresponding frequency $\omega$. With this $\omega$, the optimal controls $u_x = \sin(\omega t + \phi)$, $u_y = -\cos(\omega t + \phi)$, allow us to go to an element of the desired orbit.
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3. Step 3: The phase $\phi$ is chosen to meet the requirement on the phase of the off diagonal element, i.e., to move inside the orbit to reach the desired final value, $X_f$. 
Features of the Method

- It gives a description of the **reachable set** in a lower dimensional (orbit) space (the unit disc) and it is based on this description.
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- It gives a description of the reachable set in a lower dimensional (orbit) space (the unit disc) and it is based on this description.
- It separates the problem of finding the minimum time control in two steps: 1) **find** the frequency $\omega$ to move to the desired ‘orbit’ in the unit disc 2) **move** inside the orbit by selecting the appropriate phase parameter $\phi$. 
Extensions

Extension 1: \( K \rightarrow P \) problems for quantum systems
Extension 2: Systems with Drift
Extension 3: \( N \) Independent Qubits Controlled Simultaneously
Extension 1: $K - P$ problems for quantum systems

Extension 2: Systems with Drift

Extension 3: $N$ Independent Qubits Controlled Simultaneously
Consider a (matrix) Lie group $M$ with Lie algebra $\mathcal{L}$ with a Cartan decomposition $\mathcal{L} = \mathcal{K} \oplus \mathcal{P}$, with

$$[\mathcal{K}, \mathcal{K}] \subseteq \mathcal{K}, \quad [\mathcal{K}, \mathcal{P}] \subseteq \mathcal{P}, \quad [\mathcal{P}, \mathcal{P}] \subseteq \mathcal{K}.$$
**K — P Problems**

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The **K-P Problem** is to steer $1$ to $X_f \in M$ for $\|u\| \leq 1$ for a system

$$\dot{X} = \sum_j B_j X u_j,$$

where the $B_j$’s form an orthonormal basis of $\mathcal{P}$. 

**Extension 1:** $K - P$ problems for quantum systems

**Extension 2:** Systems with Drift

**Extension 3:** $N$ Independent Qubits Controlled Simultaneously
Quantum $K - P$ Problems

For $n$-level Quantum Systems the Lie group of interest is $SU(n)$ and the Lie algebra is $su(n)$. 
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Quantum $K - P$ Problems

For $n$-level Quantum Systems the Lie group of interest is $SU(n)$ and the Lie algebra is $su(n)$. From Cartan’s classification of symmetric spaces it follows that (up to conjugacy) there are only a finite number of $K - P$ decompositions for $su(n)$. The most interesting decompositions from the point of view of quantum control are decompositions of type $\text{AIII}$

$$
\mathcal{K} := \text{span} \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right\} \quad \mathcal{P} := \text{span} \left\{ \begin{pmatrix} 0 & C \\ -C^\dagger & 0 \end{pmatrix} \right\}
$$
Examples of $K - P$ structures in Quantum Control

Many systems of interest in Quantum Control have a $K - P$ structure. In particular, if one assumes to be able to control transitions between different energy levels, a $K - P$ structure emerges if the energy diagram is a bipartite graph.

Figure: Energy Level Diagram for Quantum Bit (a), Lambda (b), Double Lambda (c), and V (d) systems.
Symmetries

Since $[\mathcal{P}, \mathcal{K}] \subseteq \mathcal{P}$, the connected (compact) Lie group group associated with $\mathcal{K}$ ($e^{\mathcal{K}}$) acting on $SU(n)$ by conjugation is a group of symmetries for the problem.
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$$\forall B_j, \quad K \in e^{\mathcal{K}}, \quad KB_jK^{-1} = \sum_k a_{kj}B_k \in \mathcal{P}$$
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The problem on $SU(2)$ is a $K - P$ problem with $\mathcal{K} =$ diagonal matrices, $\mathcal{P} =$ anti-diagonal matrices.
Features of the K-P Problem

In application of the Pontryagin Maximum Principle, the optimal candidates have an explicit expression as

\[ X(t) = e^{-A_k t} e^{(A_k + A_p) t}, \]

with \( A_k \in \mathcal{K} \), and \( A_p \in \mathcal{P} \). In particular they are analytic.
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The problem of optimal control reduces to the problem of finding matrices \( A_k \in \mathcal{K} \) and \( A_p \in \mathcal{P} \) and time \( t \) so that \( e^{-A_k t} e^{(A_k + A_p) t} \) is the desired value and \( t \) is the minimum.
The role of Symmetry

Symmetry reduction is used to reduce the number of parameters in the search for $A_k$ and $A_p$ in $e^{-A_k t} e^{(A_k+A_p)t}$.
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The fact that the candidate optimal in the $SU(2)$ problem were explicitly integrable is a special case of the explicit expression of optimal candidate formulas in the general $K - P$ problems. The search was reduced to the search for only one parameter using symmetry reduction.
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The fact that the candidate optimal in the $SU(2)$ problem were explicitly integrable is a special case of the explicit expression of optimal candidate formulas in the general $K - P$ problems. The search was reduced to the search for only one parameter using symmetry reduction. The trajectories were considered in the quotient space which was the the unit disc in the complex plane.
Optimal Geodesics in the $SU(2)$ example
Extension 2: Systems with Drift
Systems with drift

Consider the same minimum time optimal control problem but for a system with (an orthogonal) drift.
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\[ \dot{X} = -i\sigma_z X - iu_x \sigma_x X - iu_y \sigma_y X, \quad X(0) = 1. \]
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Schrödinger operator equation with drift

\[
\dot{X} = -i\sigma_z X - iu_x \sigma_x X - iu_y \sigma_y X, \quad X(0) = 1.
\]

Make a change of variables \( Y = e^{-i\sigma_z t} X \). The equation for \( Y \) has the same form as previously considered. The reachable set at time \( t \) for the system without drift, \( R(t) \), is related to the reachable set for the system with drift, \( R_X(t) \), by \( R(t) = e^{-i\sigma_z t} R_X(t) \)
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**Schrödinger operator equation with drift**

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Make a change of variables \(Y = e^{-i\sigma_z t}X\). The equation for \(Y\) has the same form as previously considered. The reachable set at time \(t\) for the system **without** drift, \(\mathcal{R}(t)\), is related to the reachable set for the system **with** drift, \(\mathcal{R}_X(t)\), by \(\mathcal{R}(t) = e^{-i\sigma_z t}\mathcal{R}_X(t)\)

So, a characterization of the reachable set for the system without drift also gives a characterization of the reachable set for the system with drift and therefore a method to find the optimal control.
Determination of the Optimal Time and Control

From $\mathcal{R}(t) = e^{-i\sigma_z t} \mathcal{R}_x(t)$, the minimum time is the smallest $t$ so that the point of the unit disc $e^{-it} P_f$, where $P_f$ is the $(1,1)$ entry of the desired final condition, is in $\pi(\mathcal{R}(t))$, where $\pi$ is the natural projection on the space of orbits.
Extension 3: $N$ Independent Qubits Controlled Simultaneously
Simultaneous time optimal control of $N = 2$ quantum bits

Assume $N = 2$ qubits (possibly with drift) to be driven in minimum time to desired final conditions $X_{f,1}$ and $X_{f,2}$. 
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Algorithm

1. Solve the optimal control problem for system 1. Let $T_1$ be the minimum time.
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1. Solve the optimal control problem for system 1. Let \( T_1 \) be the minimum time.
2. If \( X_{f,2} \in R_2(T_1) \). Then STOP. \( T_1 \) is the minimum time. Otherwise
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2. If $X_{f,2} \in \mathcal{R}_2(T_1)$. Then STOP. $T_1$ is the minimum time. Otherwise
3. Find the smallest $T > T_1$ such that $X_{f,2} \in \mathcal{R}_2(T)$. 

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4. If $X_{f,1} \in \mathcal{R}_1(T)$ STOP. $T$ is the minimum time. Otherwise
5. Set $T_1 = T$, Exchange System 1 with System 2, and go back to Step 3.
Summary

Extension 1: $K - P$ problems for quantum systems
Extension 2: Systems with Drift
Extension 3: $N$ Independent Qubits Controlled Simultaneously
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Summary

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Summary

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2. This is equivalent to a sub-Riemannian geometry length minimizing problem with symmetries.
3. In fact it is a $K - P$ problem.
4. $K - P$ problems are very common in the control of quantum systems.
5. Sub-Riemannian length minimizing problems with symmetries can be treated in the orbit space and they give an explicit solution.
References

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Notes on the Optimal Synthesis
Features of the Optimal Synthesis:

- **Diameter**: of the system: Worst case minimum time $\equiv$ longest sub-Riemannian distance.
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- **Diameter**: of the system: Worst case minimum time $\equiv$ longest sub-Riemannian distance.
- **Critical locus**: Locus of points where geodesics lose optimality $\mathcal{CR}(M)$.
- **Cut locus**: Locus of points reached by two or more geodesics $\mathcal{CL}(M)$. 
Features of the Optimal Synthesis (ctd.)

![Cut Locus and Critical Locus in the Optimal Synthesis](image)

**Figure:** Cut Locus and Critical Locus in the Optimal Synthesis
Symmetry Reduction

**Theorem:** In the presence of a group of symmetries $G$ on $M$, the optimal synthesis is the inverse image under the natural projection of the optimal synthesis on $M/G$. In particular, calling $\mathcal{CL}$, $\mathcal{CR}$, $\mathcal{R}(t)$, respectively, cut locus, critical locus, and reachable set at time $t$, we have:

$$\mathcal{R}(t) = \pi^{-1}(\pi(\mathcal{R}(t))), \quad \mathcal{CR}(M) = \pi^{-1}(\pi(\mathcal{CR}(M))),$$

$$\mathcal{CL}(M) = \pi^{-1}(\pi(\mathcal{CL}(M))).$$
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$$R(t) = \pi^{-1}(\pi(R(t))), \quad CR(M) = \pi^{-1}(\pi(CR(M))),$$

$$CL(M) = \pi^{-1}(\pi(CL(M))).$$

In the $SU(2)$ case, the whole optimal synthesis was done in the orbit space which was the unit disc in the complex plane.
Geometry of Geodesics in the $SU(2)$ example

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