# SUBSPACE CONTROLLABILITY OF QUANTUM ISING SPIN NETWORKS WITH A CENTRAL SPIN

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# Background

# Mathematical Model for Quantum Control Systems

Schrödinger equation

$$|\dot{\psi}\rangle = -iH(u)|\psi\rangle = \Big(A + \sum_j B_j u_j(t)\Big)|\psi\rangle$$

where

- $A$, $B_j$ are skew-Hermitian matrices.
- $|\psi\rangle \in \mathcal{H}$, Hilbert space.

# Controllability

Let $\mathcal{L}$ the Lie algebra generated by $A$, and $B_j$, which is called **dynamical Lie algebra**.

Let $e^{\mathcal{L}}$ the corresponding Lie group.

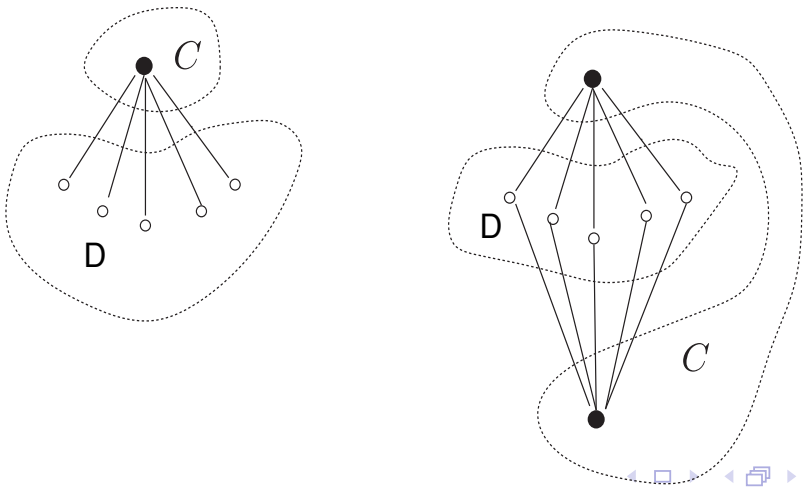
The set of states reachable from $|\psi_0\rangle$ by choosing the control fields is (dense in )

$$\{|\psi\rangle := X|\psi_0\rangle \in \mathcal{H} \,|\, X \in e^{\mathcal{L}}\}$$

If $\mathcal{L} = u(N)$ or $\mathcal{L} = su(N)$ the system is said to be **completely controllable**

# Models for Networks of Spins

**Model**: Network composed with two sets of spins: $C$ and $D$
The spins of $D$ interact via Ising interaction with the ones of set $C$, but not with each other (and viceversa).
$n_c = (n_d)$ number of spins in the set $C$ $(D)$
Interactions between each spin in $D$ and each spin in $C$ are assumed to be identical (**Symmetry**)



# Notations

**Notations**

Pauli Matrices:

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\mathbf{1}$ be the $2 \times 2$ identity matrix, we denote by:

$$S_x^n = \sum_{j=1}^{n} \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \underbrace{\sigma_x}_{j^{th}} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \in SU(2^n)$$

Similar definitions for $S_y^n$ and $S_z^n$. Moreover, we denote by

$$I_{x,y}^n = \sum_{\substack{j,k=1 \\ j \neq k}}^{n} \mathbf{1} \otimes \cdots \otimes \underbrace{\sigma_x}_{j^{th}} \otimes \cdots \otimes \underbrace{\sigma_y}_{k^{th}} \otimes \cdots \otimes \mathbf{1}$$

# Hamiltonian and Dynamical Lie algebra for the Spin Network

$$H = S_z^{n_c} \otimes S_z^{n_d} + u_x(\gamma_{n_c} S_x^{n_c} \otimes \mathbf{1} + \gamma_{n_d} \mathbf{1} \otimes S_x^{n_d}) + \\ + u_y(\gamma_{n_c} S_y^{n_c} \otimes \mathbf{1} + \gamma_{n_d} \mathbf{1} \otimes S_y^{n_d}).$$

So in our model the matrices are in $su(2^{n_c} \cdot 2^{n_d})$, we have:

$$A = -iS_z^{n_c} \otimes S_z^{n_d}$$

$$B_{x,y} = -i(\gamma_{n_c} S_{x,y}^{n_c} \otimes \mathbf{1} + \gamma_{n_d} \mathbf{1} \otimes S_{x,y}^{n_d})$$

Assuming that $|\gamma_{n_c}| \neq |\gamma_{n_d}|$ we have that the dynamical Lie algebra is

$$\mathcal{L} = \text{ Lie } \{iS_z^{n_c} \otimes S_z^{n_d},\ iS_{x,y}^{n_c} \otimes \mathbf{1},\ i\mathbf{1} \otimes S_{x,y}^{n_d}\}.$$

# Symmetries and Subspace Controllability

# Symmetries in Quantum Control

Consider again the general model for a quantum control system

$$|\dot{\psi}\rangle = -iH(u)|\psi\rangle = \Big(A + \sum_j B_j u_j(t)\Big)|\psi\rangle$$

**Def.:** A group $G$ of matrices is called **group of symmetries** if any elements of $G$ commutes with the matrices $A$ and $B_j$.

If $G$ is a group of symmetries then all the matrices in the dynamical Lie algebra $\mathcal{L}$ also commute with $G$, thus the dynamical Lie Algebra contained in $u^G(N) \subseteq u(N)$, where with $u^G(N)$ we have denoted the largest subalgebra of $u(N)$ commuting with $G$. In general the system will **not** be completely controllable.

For our spin model the group of permutation on the spin in $C$ and spin in $D$ is a symmetry group.

# Symmetries

**Theorem** (D'Alessandro, Hartwig 2018) (Application of representation theory) There exists an appropriate basis of $\mathcal{H}$, in which all the matrices of $u^G(N)$ are in a block-diagonal form. Therefore $\mathcal{H}$ splits in a set of invariant subspaces under $u^G(N)$. Furthermore on each block $\mathcal{H}_j$ $u^G(N)$ acts as $u(\dim(\mathcal{H}_j))$.

Since $\mathcal{L} \subseteq u^G(N)$, in the given basis, $\mathcal{L}$ also takes a block diagonal form.

**Definition** Let $n_j = \dim(\mathcal{H}_j)$ The system is called subspace controllable if $\mathcal{L}$ acts as $su(n_j)$ or $u(n_j)$ on each $\mathcal{H}_j$.

# Subspace Controllability

Subspace controllability allows us to do quantum computation and-or simulation on only one of the invariant subspaces $H_j$ and use the other subspaces as 'redundant dynamics' for error correction.

In order to check subspace controllability we need the decomposition of $\mathcal{H}$ and $\mathcal{L}$. We need to check how $\mathcal{L}$ acts on each invariant subspace.

In the case of the spin networks treated in this talk, the symmetry group is $S_{n_c} \otimes S_{n_d}$ where $S_N$ is the (symmetric) group of permutations of $N$ objects.

The invariant subspaces were calculated in (F.A.-D.D'Alessandro JMP 2018) and (D. D'Alessandro-J.Hartwig 2019) for $G = S_N$ and for $G = S_{n_c} \otimes S_{n_d}$ they are the tensor products of all the associated subspaces.

# Dynamical Lie algebra for a system with one or two central spins

Let $u^{S_n}$ be the largest Lie algebra in $u(2^n)$ which commutes with $S_n$. We have:

- Dim $u^{S_n} = \binom{n+3}{n}$.
- $u^{S_n} =$ is generated by $\{S_x^n,\ S_y^n,\ S_z^n,\ I_{z,z}^n\}$
- Letting $\tilde{u}^{S_n}$ be the Lie algebra generated by $\{S_x^n,\ S_y^n,\ S_z^n,\ I_{x,x}^n - I_{y,y}^n,\ I_{y,y}^n - I_{z,z}^n\}$, and letting $J^n = I_{x,x}^n + I_{y,y}^n + I_{z,z}^n$, which model the *Heisenberg interaction* for each pair of spins, we have:

$$u^{S_n}(2^n) \cap su(2^n) = \tilde{u}^{S_n}(2^n) \oplus \text{span}\{iJ^n\}.$$

- The matrix $iJ^n$ commutes with $u^{S_n}(2^n)$.

**Back to our model:**

- Group of symmetries is $G = S_{n_c} \otimes S_{n_d}$
- $\mathcal{L} \subset u^G(2^n) = iu^{S_{n_c}}(2^{n_c}) \otimes iu^{S_{n_d}}(2^{n_d})$. In fact

**Proposition:**

$$\mathcal{L} \subset \left(i\tilde{u}^{S_{n_c}} \otimes u^{S_{n_d}}\right) + \left(iu^{S_{n_c}} \otimes \tilde{u}^{S_{n_d}}\right) \subsetneq iu^{S_{n_c}} \otimes u^{S_{n_d}}.$$

# Case $n_c = 1$

In this case, we have only one central spin, thus $u^{S_1}(2) = u(2)$.

**Theorem**: The dynamical Lie Algebra $\mathcal{L}$ is given by:

$$\mathcal{L} = \Big((\text{span } \{\sigma_{x,y,z}\}) \otimes u^{S_{n_d}}(2^{n_d})\Big) \oplus \Big((\text{span } \{\mathbf{1}\}) \otimes \tilde{\mathbf{u}}^{\mathbf{S_{n_d}}}(\mathbf{2^{n_d}})\Big)$$

Although the dynamical Lie algebra is a proper subalgebra of $iu^{n_c} \otimes u^{n_d}$ subspace controllability is verified in this case

# Case $n_c = 2$ and $n_d > 2$

**Theorem**: The dynamical Lie Algebra $\mathcal{L}$ is:

$$\mathcal{G} = \left( i\tilde{u}^{S_2} \otimes u^{S_{n_d}} \right) \oplus \left( \text{span} \left( \mathbf{1} + \frac{1}{3} J^2 \right) \right) \otimes i\tilde{u}^{S_{n_d}} \oplus \left( \mathbf{1} \oplus \mathcal{S}^{n_d} \right),$$

where
$\mathcal{S}^n = Lie\{iS_x^n,\ iS_y^n,\ iS_y^n\} = \text{span}\{iS_x^n,\ iS_y^n,\ iS_y^n\}$
a $2^n \times 2^n$ representation of $su(2)$.

Although the dynamical Lie algebra is a proper subalgebra of $iu^{n_c} \otimes u^{n_d}$ subspace controllability is verified in this case as well

# Conclusions

**Some References:**

1. F. Albertini and D. D'Alessandro, Controllability of symmetric spin networks, *J. Math. Phys.* 59, 052102 (2018).
2. D. D'Alessandro and J. Hartwig, Generalized Young Symmetrizers for the Analysis of Control Systems on Tensor Spaces, arXiv:1806.01179.
3. X. Wang, D. Burgarth, and S. Schirmer, Subspace controllability of spin $\frac{1}{2}$ chains with symmetries, *Physical Review A*, **94**, 052319, (2016).
4. X. Wang, P. Pemberton-Ross, and S. Schirmer, Symmetry and Controllability for spin networks with a single-node control, *IEEE Transactions on Automatic Control*, **57**, 1945, (2012).

# Remarks

1. As compared with other treatments of subspace controllability for spin networks under symmetry our model assumes a **simultaneous** control on all spin, i.e., it does not require localized control on each spin.

2. A more general model is that multipartite systems where symmetries are present on each subset of spin.

3. The dimension of the largest invariant subspace has been found to only increase linearly with $n_d$

4. System models spin networks for instance a molecule where distances are equal among two different sets of spin.

5. Constructive controllability can be considered on each subspace.