Indirect Controllability of Quantum Systems: Lie Algebraic Techniques and Constructive Control

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Motivation

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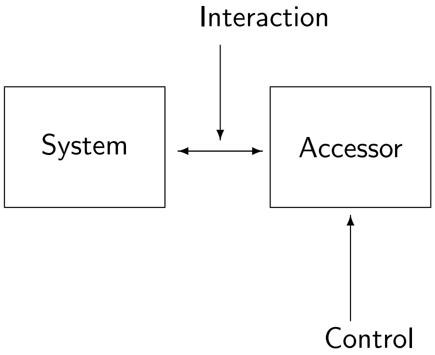
- Kurt Jacobs' work on driving quantum mechanical resonator via an auxiliary quantum bit.
- Edmond Jonckheere investigations showing that it is not possible to drive a quantum mechanical system in a de-coherence free way by direct coherent control only.
- The study of Coherent Quantum Feedback schemes where an auxiliary system is the controller.

Indirect Control Scheme



Setup:

Hamiltonian:

$$H \quad := \quad \sum_j S_j \otimes \sigma_j \quad + \quad S \otimes \mathbf{1} \quad + \quad \mathbf{1} \otimes \sigma(t)$$

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Initial State:

$$\rho_{TOT}(0) := \rho_S(0) \otimes \rho_A(0)$$

Setup (ctd.):

Evolution of the system is given by

$$\rho_S(t) = Tr_A(X(t)\rho_S(0) \otimes \rho_A(0)X^{\dagger}(t)),$$

where $X(t)$ is the solution of the Schrödinger equation

$$i\dot{X} = HX, \qquad X(0) = \mathbf{1},$$

for the total system $S + A$ (System plus Accessor). Tr_A is the partial trace with respect to A.

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Indirect Controllability Questions:

What is the set of states that can be reached for S with this scheme? How can we drive S indirectly (constructive controllability)?

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Indirect Controllability is defined as the possibility of inducing any unitary evolution on the state ρ_S of S with the coherent quantum feedback scheme.

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Dynamical Lie Algebra: The set of available evolutions for the total system is the Lie group $e^{\mathcal{L}}$ associated with the dynamical Lie algebra $\mathcal{L}$ generated by the allowed Hamiltonians (as the control changes with time).

If $\mathcal{L} = su(n_S n_A)$, then the system $S + A$ is **completely controllable**, i.e., every unitary evolution on ρ_{TOT} can be performed. **Complete controllability** implies **indirect controllability**.

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The set of available states for S is given by

$$\mathcal{F} := Tr_A\{U\rho_S(0) \otimes \rho_A(0)U^\dagger \mid U \in e^{\mathcal{L}}\}$$

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- The initial state of the auxiliary system ρ_A.

Question: Under what conditions on $\mathcal{L}$ and ρ_A we have indirect controllability?

Assumption 1: There exists full control on the auxiliary system A, i.e., $\mathcal{L}$ contains the full space of matrices

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Assumption 2: The dynamical Lie algebra $\mathcal{L}$ contains at least one element of the form $S \otimes \sigma$ (i.e., there is a nontrivial interaction between S and A.

Theorem 1 Assume the dimension of system A, n_A, is ≥ 3, then indirect controllability given $\rho_A(0)$ is verified if and only if $\mathcal{L} = su(n_S n_A)$, i.e., the system $S + A$ is completely controllable, independently of ρ_A.

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Let $\mathcal{K}$ the space of matrices K such that $K \otimes \mathbf{1} \in \mathcal{L}$ and let $\mathcal{P}$ the space of matrices P such that $iP \otimes \sigma$ is in $\mathcal{L}$, ($\sigma \in su(2)$). Define the Lie algebra

$$\mathcal{L}_S := \mathcal{K} \oplus \mathcal{P}$$

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Theorem 2 Assume $n_A = 2$. System S is indirectly controllable given ρ_A if and only if one of the following two situations occurs:

1. $\mathcal{L} = su(n_{S n_A})$, i.e., the system $S + A$ is completely controllable.
2. ρ_A is a pure state and $\mathcal{L}_S = u(n_S)$.

Theorem 3 Consider the subspace

$$\mathcal{V} := \bigoplus_{k=0}^{\infty} ad_{\mathcal{L}}^{k} \left(\mathrm{span}\{i\rho_S \otimes \rho_A\}\right),$$

If every $X\rho_S X^\dagger$ is reachable, then, $Tr_A(\mathcal{V}) = u(n_S)$.

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Examples exist showing that this condition of controllability is in general not sufficient.

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4. We study the structure of $\mathcal{P}$ starting from a Cartan subalgebra $\mathcal{A}$ of $\mathcal{P}$, and prove that if there is indirect controllability $\mathcal{L}_S = u(n_S)$. Otherwise transformations on S will take a block diagonal form.

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5. This, if $n_A \geq 3$ implies $\mathcal{L} = su(n_S n_A)$.

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Proof (Theorem 2) $(n_A = 2)$ $\mathcal{L}_S = u(n_S)$ and ρ_A pure implies indirect controllability of S

Let $\tilde{\mathcal{P}}$ the subspace of matrices in $\mathcal{P}$ with zero trace and $\mathcal{A}$ a maximal Abelian subalgebra in $\tilde{\mathcal{P}}$.
The Lie group $e^{\mathcal{L}}$ contains all elements of the form:

$$K \otimes \mathbf{1}, \ \mathbf{1} \otimes B, \ e^{itA \otimes \bar{\sigma}},$$

with $K \in e^{\mathcal{K}}$, $B \in SU(2)$ and $A \in \mathcal{A}$, $\bar{\sigma} := \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$.

Proof (ctd.)
Since $i\bar{\sigma}E_1 = iE_1\bar{\sigma} = E_1$, for a general matrix ρ we have:

$$e^{itA\otimes\bar{\sigma}}\left(\rho\otimes E_1\right)e^{-itA\otimes\bar{\sigma}} = \left(e^{tA}\rho e^{-tA}\right)\otimes E_1$$

Let $Z_i = K_i\otimes\mathbf{1}$, and $X = e^{iA\otimes\bar{\sigma}}$, then $Z_i,\ X\in e^{\mathcal{L}}$. We have:

$$Z_1XZ_2\left(\rho_1\otimes E_1\right)\left(Z_1XZ_2\right)^\dagger =$$

$$K_1AK_2\rho_1(K_1AK_2)^\dagger\otimes E_1 = \rho_2\otimes E_1.$$

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The result follows from Cartan decomposition of $SU(n_S)$ and by taking the partial trace.

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1. Call γ the map $\rho_S \rightarrow Tr_A(U\rho_S \otimes \rho_A U^\dagger$, which depends on ρ_A. If ρ_A is mixed (convex combination of maps)

$$c_1\gamma_1(\rho_S) + c_2\gamma_2(\rho_S) = X\rho_S X^\dagger.$$

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3. From this
$$\frac{1}{2}(\gamma_1[\rho_S]+\gamma_2[\rho_S]) = \frac{1}{2}Tr_A\left(U\rho_S \otimes \rho_{A,1}U^\dagger\right)+\frac{1}{2}Tr_A\left(U\rho_S \otimes \rho_{A,2}U^\dagger\right)$$
$$= Tr_A\left(U\rho_S \otimes (\frac{1}{2}\mathbf{1})U^\dagger\right) = X\rho_S X^\dagger,$$
for every ρ_S.

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for every ρ_S.
4. Therefore $\frac{1}{2}\mathbf{1}$ has the indirect controllability property which implies $\mathcal{L} = su(n_S n_A)$.

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with ω_S and J constant, and $\sigma_{x,y,z}$ Pauli matrices and $u_{x,y,z}$ components of the (control) electro-magnetic field.

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with ω_S and J constant, and $\sigma_{x,y,z}$ Pauli matrices and $u_{x,y,z}$ components of the (control) electro-magnetic field.

The dynamical Lie algebra is

$$\mathcal{L} := \mathtt{span}\{\sigma_z \otimes \mathbf{1},\, i\{\sigma_x, \sigma_y\} \otimes \sigma_1,\, \mathbf{1} \otimes \sigma_2 \mid \sigma_1,\, \sigma_2 \in su(2)\},$$

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Therefore

$$\mathcal{L}_S = \mathcal{K} \oplus \mathcal{P} = su(2).$$

Constructive Control Algorithm for the indirect control of qubit S through qubit A.

Main ideas:

1) Without loss of generality, the initial state $\rho_A(0)$ can be assumed equal to $\rho(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

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3)

$$e^{t(\omega_S\sigma_z \otimes 1 \pm iJ\sigma_x \otimes \sigma_z)} \rho_S \otimes E_1 e^{-t(\omega_S\sigma_z \otimes 1 \pm iJ\sigma_x \otimes \sigma_z)} =$$

$$\left(e^{t(\omega_S\sigma_z \pm J\sigma_x)} \rho_S e^{-t(\omega_S\sigma_z \pm J\sigma_x)} \right) \otimes E_1.$$

Main Ideas (ctd.)

4) Iterating the above evolutions alternating $\omega_S \sigma_z \otimes 1 \pm iJ\sigma_x \otimes \sigma_z)$, we obtain an alternating of $\omega_S \sigma_z \pm J\sigma_x$ and $\omega_S \sigma_z - J\sigma_x$ on S.

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5) The control is obtained by a generalized Euler decomposition technique on $SU(2)$.

References

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