On $K - P$ sub-Riemannian Problems and their Cut-Locus

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Outline

1. $K - P$ sub-Riemannian problems
2. Symmetry Reduction
3. Geometry of the Quotient Space
4. Application: Determination of the cut locus
5. Conclusions
$K - P$ sub-Riemannian problems
K – P Decomposition

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**Definition**

A $K - P$ decomposition of $\mathfrak{g}$, is a decomposition $\mathfrak{g} = \mathcal{K} \oplus \mathcal{P}$ into subspaces orthogonal with respect to the Killing form and with:

1. $[\mathcal{K}, \mathcal{K}] \subseteq \mathcal{K}$
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3. $[\mathcal{P}, \mathcal{P}] \subseteq \mathcal{K}$
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Example of $K - P$ Decomposition

Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{su}(n)$ with Killing form $B(A, C) := \frac{1}{n} \text{Tr}(AC^\dagger)$ and choose two positive integers $p$ and $q$ such that $p + q = n$. 
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\[ P := K^\perp; \]

The conditions:

\[ [K, K] \subseteq K, \quad [K, P] \subseteq P, \quad [P, P] \subseteq K \]

are verified.
The $K - P$ problem in time-optimal control

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- Let $\{B_j\}$ be an orthonormal basis for $\mathcal{P}$.
- $\{u_j\}$ real-valued control functions depending upon time with $\sum_j u_j^2(t) \leq 1$. 

Let a trajectory $U(t)$ on $G$ evolve according to:

$$\dot{U}(t) = \sum_{j=1}^{\infty} u_j(t) B_j U(t), \quad U(0) = 1_{K}$$ 

Sub-Riemannian Optimal Control Problem

Given a desired final condition $X_f$, find the controls and trajectory which reaches $X_f$ in minimum time.
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$$\dot{U}(t) = \sum_{j=1}^{m} u_j(t) B_j U(t), \quad U(0) = 1$$  \hspace{1cm} (1)
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**$K - P$ sub-Riemannian Optimal Control Problem**

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**Sub-Riemannian Geometry Interpretation**

The time optimal control problem can be interpreted as the problem of finding the length minimizing sub-Riemannian geodesics of $G$ where the sub-Riemannian structure is specified by a set of vector fields $\Delta := \{B_j U, \; j = 1, \ldots, m\}$.
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Related question

Characterize the cut-locus: Locus of points where the sub-Riemannian geodesics lose optimality.
Example: Time Optimal Control of a Quantum Bit

Consider a two level quantum system with Hamiltonian $H = u_x \sigma_x + u_y \sigma_y$, where $\sigma_x$ and $\sigma_y$ are the Pauli matrices subject to a control field $u_x, u_y$ bounded in magnitude, i.e., $u_x^2 + u_y^2 \leq 1$.

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
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Optimal control problem

For the Schrödinger operator equation

$${\dot{U}} = -iu_x \sigma_x U - iu_y \sigma_y U, \quad U(0) = 1,$$

given a final condition $X_f \in SU(2)$, find the time optimal control with $\|u\|^2 \leq 1$ driving the state $U$ from $1$ to $X_f$, in minimum time.
Motivations

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- In applications to Quantum Computation minimum time is a natural requirement.
- Fast dynamics is a way to counteract the effect of the environment (decoherence)
- The trade-off between energy and time characterizes the Quantum Speed Limit and gives the bound in the time-energy uncertainty relations (cf., e.g., M.M. Taddei Thesis (2014) arxiv.org/pdf/1407.4343).
Symmetry Reduction
Conjugacy Action

Let $K := e^K$ the Lie group associated with the Lie subalgebra $\mathcal{K}$ in the $K - P$ decomposition.
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$K$ acts on $G$ by conjugation ($k \in K$, $U \in G$)

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As a consequence of $[\mathcal{K}, \mathcal{P}] \subseteq \mathcal{P}$, $kBk^{-1} \in \mathcal{P}$. The action is in fact an isometry.
Consequences for the Optimal Control Problem

\(\gamma\) is an optimal sub-Riemannian geodesic joining 1 to \(X_f\) if and only if \(k\gamma k^{-1}\) is an optimal geodesic joining 1 to \(kX_f k^{-1}\) for every \(k \in \mathcal{K}\).
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Reduction

This suggests to study the optimal control problem in the quotient space \( G/K \), where the equivalence class of \( x \in G \) is
\( \pi(x) = \{ y \in G | kyk^{-1} = x \text{ for some } k \in K \} \). (Here \( \pi : G \rightarrow G/K \) is the natural projection)
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Reduction Theorem (Albertini and D.D. JDCS 2018)

1) sub-Riemannian optimal geodesics are inverse images (under natural projection $\pi$) of trajectories in $G/K$. 2) Reachable sets at any time are inverse images of sets in $G/K$. 3) The cut locus in $G$ is the inverse image of a set in $G/K$. 

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Geometry of the Quotient Space
G/K as a stratified space

G/K has the structure of a Stratified Space
$G/K$ as a stratified space

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Stratification is obtained via **Orbit Type Decomposition**
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Definition
Let $K_x$ be the isotropy group of $x \in G$, and say that $x, y \in G$ have the same orbit type if there exists a $k \in K$ such that $kK_xk^{-1} = K_y$. The set of isotropy groups conjugate to a subgroup $H$ is denoted by $(H)$. The set of elements of $G$ with isotropy group $(H)$ (i.e., with isotropy type $(H)$) is denoted by $G_{(H)}$. 
$G/K$ as a stratified space

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**Definition**

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**Fact**

Points in the same orbit belong to the same orbit type $G_{(H)}$. Therefore it makes sense to define the quotients $G_{(H)}/K$. 
Fact 1

\( G(H)/K \) induces a stratification of \( G/K \) where the strata are given by the connected components of \( G(H)/K \) for each isotropy type \( (H) \).
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Fact 2

There exists a minimal isotropy type \(H_{min}\) such that \((H_{min})\) contains a subgroup of a group in \((H)\) for every isotropy type \((H)\). The associated orbit type \(G_{H_{min}}/K\) is an open and dense manifold in \(G/K\). The preimage \(\pi^{-1}(G_{H_{min}}/K)\) is an open and dense submanifold of \(G\). It is called the regular part of \(G\), \(G_{reg} = \pi^{-1}(G_{H_{min}}/K)\). \(G_{reg}/K = G_{(H_{min})}/K\). \(G - G_{reg} = G_{sing}\) is called the singular part of \(G\).
Example: $SU(2)$

For the conjugation action of diagonal matrices $K$ in $SU(2)$ on $SU(2)$, there are only two possible isotropy groups (types) $H_{min} \leq H$:

\[ H_{min} = \{ \pm 1 \}, \quad H = K \text{ itself.} \]
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Reduction to the Unit Disc in the Complex Plane

For $U \in SU(2)$

\[ U := \begin{pmatrix} x & y \\ -y^* & x^* \end{pmatrix}, \quad |x|^2 + |y|^2 = 1 \]

the action of (diagonal) $D$, $U \rightarrow DUD^\dagger$ only affects the off-diagonal elements. The complex number $x$ (an element of the unit disc in the complex plane) determines the equivalence class (orbit). The interior (boundary) of the disc is the regular (singular) part of $G/K$. 
Metric on the Quotient Space

**Question:**

We would like to define a metric on $G/K$ so that sub-Riemannian geodesics on $G$ correspond to Riemannian geodesics on $G/K$, with the same length. Since $G/K$ is not a manifold we will define a metric on $G_{reg}/K$. 

Theorem (D.D. and B. Sheller)

Assume $H_{\text{min}}$ is discrete. Then for each $x \in G$ the map $\varpi : \mathfrak{g}_{x} \rightarrow \mathfrak{p}$ is an isomorphism. The metric (for $V, W \in \mathfrak{t}_{\varpi(x)}G_{reg}/K$) $g_{\varpi(x)}(V, W) = B(R_{x}1 \varpi_{1}V, R_{x}1 \varpi_{1}W)$, where $B$ is the Killing form on $\mathfrak{g}$ restricted to $\mathfrak{p}$, i.e., does not depend on the representative $x$ in the orbit $\varpi_{1}(\varpi(x))$. 

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Theorem (D.D. and B. Sheller)
Assume $H_{min}$ is discrete. Then for each $x \in G$ the map

$$\pi_* : R_{x_*} \mathcal{P} \to T_{\pi(x)} G_{reg}/K$$

is an isomorphism. The metric (for $V, W \in T_{\pi(x)} G_{reg}/K$)

$$g_{\pi(x)}(V, W) := B(R_{x^{-1}_*}\pi_*^{-1}V, R_{x^{-1}_*}\pi_*^{-1}W),$$

where $B$ is the Killing form on $g$ restricted to $\mathcal{P}$, is well defined, i.e., does not depend on the representative $x$ in the orbit $\pi^{-1}(\pi(x))$. 
Sub-Riemannian geodesics on $G$ vs Riemannian geodesics on $G/K$

**Theorem (D.D. and B. Sheller)**

Assume $\gamma = \gamma(t)$ is a sub-Riemannian geodesic defined in $[0, T]$ optimally connecting $1$ and $q \in G_{\text{reg}}$. Then $\pi(\gamma)$ is a Riemannian geodesic from $\pi(\gamma(t_0))$ to $\pi(\gamma(T)) = \pi(q)$, for any $t_0 \in (0, T)$.

Moreover

$$\lim_{t_0 \to 0^+} d_Q(\pi(\gamma(t_0)), \pi(q)) = d(1, q).$$

where $d_Q$ is the Riemannian distance on $G/K$ and $d$ is the sub-Riemannian distance on $G$. 

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**Remark**

Notice that since $\pi(q)$ is a regular point, $\pi(\gamma(t))$ is a regular point for every $t \in (0, T)$ because if $\pi(\gamma(t))$ becomes singular for some time $t$ it will lose optimality at $t$ (F. Albertini, D.D. JDCS 2017)
Geodesics for the $SU(2)$ problem
Application to the Cut-Locus
Points on the Cut Locus

Points in the cut locus are of two types: 1) Singular points 2) Regular points where Riemannian geodesics lose optimality.
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Theorem D.D. and B. Sheller

Suppose that the sectional curvature of $G_{\text{reg}}/K$ under the given metric is nonpositive and that $G_{\text{reg}}/K$ is simply connected. Then the intersection of the cut locus with $G_{\text{reg}}$ is empty.
Determination of the cut locus for the $SU(2)$ example

On the disk $SU(2)/K$ the sectional curvature is calculated to be

$$\hat{k} := \frac{-2}{1 - r^2} < 0.$$
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Conclusions
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2. They admit a group of symmetries $K$ which allows us to reduce the problem on a quotient space $G/K$.
3. We have introduced a Riemannian metric on $G/K$ which allows us to study sub-Riemannian geodesics on $G$ as Riemannian geodesics on $G/K$. 

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K – P sub-Riemannian Problems

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Summary

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2. They admit a group of symmetries $K$ which allows us to reduce the problem on a quotient space $G/K$.
3. We have introduced a Riemannian metric on $G/K$ which allows us to study sub-Riemannian geodesics on $G$ as Riemannian geodesics on $G/K$.
4. As an application we can determine the cut locus as the singular part of the orbit space decomposition associated with the action of $K$ on $G$. We have done this in the $SU(2)$ case.