Algebra Universalis, **24** (1987) 101–110

0002–5240/87/020101–10\$01.50 + 0.20/0

Saturated algebras in filtral varieties

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A category is said to be *balanced* if every morphism that is mono and epic is an isomorphism. As any variety of algebras can be viewed as a category, one can investigate balanced varieties. However, balanced varieties seem to be few and far between. One can localize the question and consider those algebras $\mathbf{A}$ such that every injective epimorphism from $\mathbf{A}$ to $\mathbf{B}$ in the variety is an isomorphism. Such algebras have been called *saturated* in the literature.

In this paper we characterize all balanced filtral varieties and provide some necessary and some sufficient conditions for an algebra to be saturated. However we do not quite complete the characterization of saturated algebras in filtral varieties.

We assume the reader is familiar with the basic notions of universal algebra. Consult the text [BS] for definitions not supplied here.

Let $\mathbf{A}$ be a subalgebra of a direct product $\Pi\,(\mathbf{A}_i : i \in I)$, and let F be a filter on I. Then F induces a congruence, η_F, on $\mathbf{A}$, given by

$$a \equiv b \pmod{\eta_F} \text{ if and only if } [\![a = b]\!] \in F$$

where

$$[\![a = b]\!] = \{i \in I : a_i = b_i\}.$$

A variety $\mathscr{V}$ is called *filtral* if, for every $\mathbf{A}$ in $\mathscr{V}$, and every representation of $\mathbf{A}$ as a subdirect product of subdirectly irreducible algebras, every congruence on $\mathbf{A}$ is of the form η_F for some filter F.

An important subclass of the filtral varieties are the *discriminator varieties*. Let S be any set. We define the *normal transform* on S to be the quaternary function n^S from S^4 to S defined by $n^S(a, b, c, d) :=$ if $a = b$ then c, else d. $\mathscr{V}$ is a *discriminator variety* if there is a term $t(x, y, u, v)$ in the language of $\mathscr{V}$ such that, for every $\mathbf{A} \in \mathscr{V}$, $\mathbf{A}$ is subdirectly irreducible if and only if $t^{\mathbf{A}} = n^A$. It is

Presented by Stanley Burris. Received May 20, 1986. Accepted for publication in final form March 20, 1987.

well-known that every discriminator variety is filtral and that every filtral variety is congruence distributive, semi-simple and has the congruence extension property. For details and references see [K] or [W].

Most of this paper deals with a fixed filtral variety $\mathscr{V}$. We let $\mathscr{M}$ denote the class of simple members of $\mathscr{V}$ and $\mathscr{M}^+$ the class of algebras that are either simple or trivial. $\mathscr{M}^+$ is closed under the formation of ultraproducts, subalgebras and isomorphic images, that is $\mathscr{M}^+ = ISP_U(\mathscr{M}^+)$. For $\mathbf{A}$ in $\mathscr{V}$, we define a topological space $\operatorname{Spec}(\mathbf{A})$ as follows: $\operatorname{Spec}(\mathbf{A})$ consists of all $\theta \in \operatorname{Con}(\mathbf{A})$ (the congruence lattice of $\mathbf{A}$) such that $\mathbf{A}/\theta \in \mathscr{M}^+$. Notice that $\mathbf{A} \leqslant \Pi(\mathbf{A}/\theta : \theta \in \operatorname{Spec}(\mathbf{A}))$. A subbasis for the topology on $\operatorname{Spec}(\mathbf{A})$ is given by all sets of the form $[\![a = b]\!]$ and $[\![a \neq b]\!]$, as a and b range through A. With this topology, $\operatorname{Spec}(\mathbf{A})$ becomes a boolean space. In fact, we have a contravariant functor from $\mathscr{V}$ to the category of boolean spaces. If $f : \mathbf{A} \to \mathbf{B}$ is a homomorphism then $f^* : \operatorname{Spec}(\mathbf{B}) \to \operatorname{Spec}(\mathbf{A})$ is continuous, where

$$f^*(\theta) = \{(a, b) \in A^2 : (fa, fb) \in \theta\}.$$

See [K, section 2] for the proofs.

The definition of boolean product can be found in [BS, IV.8.1]. We write $\Gamma^a(\mathscr{K})$ for the class of boolean products of members of $\mathscr{K}$. Let $\mathscr{V}$ be a filtral variety and $\mathbf{A} \in \mathscr{V}$. Let $\mathbf{B} = \Pi(\mathbf{A}/\theta : \theta \in \operatorname{Spec}(\mathbf{A}))$. We have an induced function $\bar{n} : A^4 \to B$ given by

$$\bar{n}(a, b, c, d)/\theta = n^{A/\theta}(a/\theta, b/\theta, c/\theta, d/\theta) \text{ for each } \theta \in \operatorname{Spec}(\mathbf{A}).$$

Let $\bar{\mathbf{A}}$ denote the closure of $\mathbf{A}$ under $\bar{n}$. Thus $\mathbf{A} \leqslant \bar{\mathbf{A}} \leqslant \mathbf{B}$ and $f^* : \operatorname{Spec}(\bar{\mathbf{A}}) \to \operatorname{Spec}(\mathbf{A})$ is surjective (use CEP plus Zorn's lemma) where f is the inclusion of $\mathbf{A}$ in $\bar{\mathbf{A}}$. In [K], Krauss proved that f^* is actually a homeomorphism and furthermore, $\mathbf{A} = \bar{\mathbf{A}}$ iff $\mathbf{A} \in \Gamma^a(\mathbf{M}^+)$ iff $\mathbf{A}$ is a boolean product of $\Pi(\mathbf{A}/\theta : \theta \in \operatorname{Spec}(\mathbf{A}))$.

DEFINITION. Let $\mathscr{K}$ be a class of algebras.

1. A homomorphism $f : \mathbf{A} \to \mathbf{B}$ is a *$\mathscr{K}$-epimorphism* if $\mathbf{A}, \mathbf{B} \in \mathscr{K}$ and for every $\mathbf{C} \in \mathscr{K}$ and $g_i : \mathbf{B} \to \mathbf{C}$, $i = 0, 1$; $g_1 \circ f = g_0 \circ f$ implies $g_1 = g_0$.
2. An algebra $\mathbf{A}$ is *$\mathscr{K}$-saturated* if $\mathbf{A} \in \mathscr{K}$ and every injective $\mathscr{K}$-epimorphism on $\mathbf{A}$ is an isomorphism. The class of $\mathscr{K}$-saturated algebras is denoted $\mathscr{K}_{\text{sat}}$.

It is easy to see that a variety $\mathscr{V}$ is balanced if and only if every $\mathscr{V}$-epimorphism is surjective. Furthermore, it suffices to consider injective epimorphisms, for, if $f : \mathbf{A} \to \mathbf{B}$, then f is epimorphic if and only if the inclusion of $\operatorname{Im}(f)$ in $\mathbf{B}$ is epimorphic. Thus a variety $\mathscr{V}$ is balanced if and only if $\mathscr{V} = \mathscr{V}_{\text{sat}}$. It is easy to check that trivial algebras are always saturated. Let $f:\mathbf{A}\to\mathbf{B}$ be an injective homomorphism and let $\theta\in\mathrm{Con}(\mathbf{B})$. p_θ (or p_θ^B if necessary for clarity) will always denote the canonical homomorphism $\mathbf{B}\to\mathbf{B}/\theta$. If $\psi=f^*(\theta)$ on $\mathbf{A}$, then $f/\theta:\mathbf{A}/\psi\to\mathbf{B}/\theta$ is the induced embedding such that $p_\theta\circ f=f/\theta\circ p_\psi$. It is easy to check that if f is an epimorphism then f/θ is an epimorphism.

Our first theorem is based on the simple observation that, in a filtral variety, the inclusion of an algebra $\mathbf{A}$ in its closure $\bar{\mathbf{A}}$ is epimorphic.

THEOREM 1. *Let $\mathscr{V}$ be a filtral variety. Then $\mathscr{V}_{\mathrm{sat}}\subseteq\Gamma^a(\mathscr{M}^+)$.*

Proof. Let $\mathbf{A}\in\mathscr{V}$. It suffices to show that the embedding of $\mathbf{A}$ into $\bar{\mathbf{A}}$ is an epimorphism, for then $\mathbf{A}\in\mathscr{V}_{\mathrm{sat}}$ implies A closed under $\bar{n}$, so $\mathbf{A}$ is a boolean product of $\Pi(\mathbf{A}/\theta:\theta\in\mathrm{Spec}(\mathbf{A}))$. Let $\mathbf{B}\in\mathscr{V}$, g and h maps from $\bar{\mathbf{A}}$ to $\mathbf{B}$ and suppose $g\restriction A=h\restriction A$. Now $\bar{A}$ can be thought of as the subuniverse of $\Pi(\mathbf{A}/\theta:\theta\in\mathrm{Spec}(\mathbf{A}))$ generated by A in the language of $\mathscr{V}$ expanded to include a new 4-ary operation symbol, $\bar{n}$. As g and h agree on the generating set, to show $g=h$, it suffices to show that they are homomorphisms in the expanded language. For this, we need only verify that $g(\bar{n}^{\bar{A}}(a,b,c,d))=\bar{n}^{B}(ga,gb,gc,gd)$.

Let $\beta\in\mathrm{Spec}(\mathbf{B})$ and $\alpha=g^*(\beta)\in\mathrm{Spec}(\bar{\mathbf{A}})$. If $g(a)\equiv g(b)(\mathrm{mod}\,\beta)$ then $a\equiv b(\mathrm{mod}\,\alpha)$, and by the definition of $\bar{n}$, $\bar{n}^{\bar{A}}(a,b,c,d)\equiv c\ (\mathrm{mod}\,\alpha)$, so $g(\bar{n}^{\bar{A}}(a,b,c,d))\equiv g(c)\equiv\bar{n}^{B}(ga,gb,gc,gd)\ (\mathrm{mod}\,\beta)$. Similarly, $g(a)\not\equiv g(b)$ *(mod β)* implies $g(\bar{n}^{\bar{A}}(a,b,c,d))\equiv g(d)\equiv\bar{n}^{\bar{B}}(ga,gb,gc,gd)(\mathrm{mod}\,\beta)$. Since $\mathrm{Spec}(\mathbf{B})$ includes all completely meet-irreducible congruences on $\mathbf{B}$, our equality must hold. ●

THEOREM 2. *Let $\mathscr{V}$ be a filtral variety. Then $\Gamma^a(\mathscr{M}^+_{\mathrm{sat}})\subseteq\mathscr{V}_{\mathrm{sat}}$.*

Proof. Let $\mathbf{A}\in\Gamma^a(\mathscr{M}^+_{\mathrm{sat}})$. $\mathbf{A}=\bar{\mathbf{A}}$ is a boolean product of $\Pi(\mathbf{A}/\theta:\theta\in\mathrm{Spec}(\mathbf{A}))$, in which for all $\theta\in\mathrm{Spec}(\mathbf{A})$, $\mathbf{A}/\theta\in\mathscr{M}^+_{\mathrm{sat}}$. (As remarked above, a trivial algebra is always saturated.)

Let $f:\mathbf{A}\to\mathbf{B}$ be an injective epimorphism. We must show f is surjective. As mentioned above, we have a continuous map $f^*:\mathrm{Spec}(\mathbf{B})\to\mathrm{Spec}(\mathbf{A})$. Let $\theta\in\mathrm{Spec}(\mathbf{B})$. We claim first that the induced map f/θ is an isomorphism. Set $\psi=f^*(\theta)$. Then $\mathbf{A}/\psi\in\mathscr{M}^+_{\mathrm{sat}}$. But $f/\theta\circ p_\psi^A=p_\theta^B\circ f$ is an epimorphism, so f/θ is an epimorphism. Therefore, since $\mathbf{A}/\psi$ is saturated, f/θ is an isomorphism.

Next we verify that f^* is injective. Suppose that $f^*(\theta)=f^*(\tau)=\psi$. Then $p_\theta^B\circ f=f/\theta\circ p_\psi^A$ and $p_\tau^B\circ f=f/\tau\circ p_\psi^A$ and from above f/θ and f/τ are invertible. Thus $p_\theta^B\circ f=f/\theta\circ(f/\tau)^{-1}\circ p_\tau^B\circ f$, so, since f is injective, hence monic, $p_\theta=f/\theta\circ(f/\tau)^{-1}\circ p_\tau$. Taking the kernels of both sides, $\theta=\tau$.

By the CEP and the injectivity of f, f^* is surjective. Thus f^* is a continuous bijection of boolean spaces, so by the Stone Representation Theorem, the dual boolean algebras are isomorphic, and therefore f^* is a homeomorphism. Now we can argue that f is surjective. Fix $b \in B$. $\mathrm{Spec}\,(\mathbf{B}) = \bigcup \{[\![b = f(a)]\!] : a \in A\}$. (Proof: for $\theta \in \mathrm{Spec}\,(\mathbf{B})$, let $\psi = f^*(\theta)$ and let a be any element of A such that $f/\theta \circ p_\psi(a) = p_\theta(b)$.) By compactness, there are $a_1, a_2, \ldots, a_k \in A$ such that $\mathrm{Spec}\,(\mathbf{B}) = \bigcup \{[\![b = f(a_j)]\!] : j = 1, 2, \ldots, k\}$. Since f^* is a homeomorphism, $f^*([\![b = f(a_j)]\!]) = N_j$ is a clopen subset of $\mathrm{Spec}\,(\mathbf{A})$. Finally since $\mathbf{A}$ is a boolean product, we can apply the patchwork property to find $a \in A$ such that $[\![a = a_j]\!] \supseteq N_j$ for $j = 1, \ldots, k$. Then $f(a) = b$, since if $\theta \in \mathrm{Spec}\,(\mathbf{B})$, we have $\theta \in [\![b = f(a_j)]\!]$, for some $j \leqslant k$, so $f^*(\theta) \in N_j$. But then $a \equiv a_j \pmod{f^*(\theta)}$, so $f(a) \equiv f(a_j) \equiv b \pmod{\theta}$. Thus f is surjective. ●

We can use Theorems 1 and 2 to characterize all balanced filtral varieties. A special case of this corollary was discovered independently by S. Comer [C].

COROLLARY 3. (i) *A balanced filtral variety is a discriminator variety.*
(ii) *Let $\mathcal{V}$ be a discriminator variety. The following are equivalent*:
(a) *$\mathcal{V}$ is balanced*
(b) $\mathcal{M} \subseteq \mathcal{V}_{\mathrm{sat}}$
(c) $\mathcal{M} \subseteq \mathcal{M}^+_{\mathrm{sat}}$

Proof. By Theorem 1, if $\mathcal{V}$ is a balanced filtral variety, then $\mathcal{V} = \Gamma^a(\mathcal{M}^+)$, that is, $\mathcal{V}$ is boolean representable. By [KC, 6.16] $\mathcal{V}$ is a discriminator variety. For (ii), every discriminator variety is boolean representable by members of $\mathcal{M}^+$, so (a) follows from (c) by Theorem 2. Also (a)$\Rightarrow$(b) a fortiori. To check (b)$\Rightarrow$(c) it suffices to verify that if $f : \mathbf{A} \to \mathbf{B}$ is an $\mathcal{M}^+$-epimorphism, then f is a $\mathcal{V}$-epimorphism.

Suppose $\mathbf{C} \in \mathcal{V}$ and $g_i : \mathbf{B} \to \mathbf{C}$ for $i = 1, 2$. If $g_1 \neq g_2$ then there is $b \in B$ such that $g_1(b) \neq g_2(b)$. Therefore, there is a simple algebra $\mathbf{D} \in \mathcal{M}$ and a surjective map $p : \mathbf{C} \to \mathbf{D}$ such that $p \circ g_1(b) \neq p \circ g_2(b)$. Since f is assumed to be an $\mathcal{M}^+$-epimorphism, we must have $p \circ g_1 \circ f \neq p \circ g_2 \circ f$ and therefore $g_1 \circ f \neq g_2 \circ f$. Thus f is a $\mathcal{V}$-epimorphism. ●

EXAMPLES. 1. Let $\mathcal{BA}$ be the variety of boolean algebras. The well-known fact that $\mathcal{BA}$ is balanced follows from Corollary 3 since both the one and two element boolean algebras are saturated in $\mathcal{M}^+$. Observe also that despite the Corollary, it is not true that $\mathcal{V}$ balanced implies $\mathcal{M}^+$ balanced since in this example, the map from the two element boolean algebra to the trivial algebra is both mono and epic in $\mathcal{M}^+$.

2. Let n be a positive integer and let $\mathcal{V}$ be the variety of rings satisfying $x^n = x$. Then $\mathcal{V}$ is a discriminator variety and $\mathcal{M}$ is the collection of all finite fields

whose order divides n (see [MW].) If $\mathbf{E}$ is a proper subfield of $\mathbf{F} \in \mathcal{M}$, then the Galois group of $\mathbf{F}$ over $\mathbf{E}$ has order equal to $[\mathbf{F}:\mathbf{E}] > 1$. Therefore the inclusion of $\mathbf{E}$ into $\mathbf{F}$ is not epimorphic, so $\mathbf{E}$ is saturated. Therefore, by Corollary 3, $\mathbf{V}$ is balanced.

3. Let $\mathscr{D}$ be the variety of distributive lattices. Then $\mathscr{D}$ is filtral, but not a discriminator variety, so $\mathscr{D}$ is not balanced. The only simple member of $\mathscr{D}$ is the two element lattice, $\mathbf{2}$, which is therefore $\mathcal{M}^+$-saturated. Combining Theorems 1 and 2 we conclude that $\mathscr{D}_{\text{sat}} = \Gamma^a\{\mathbf{1}, \mathbf{2}\}$. This latter class is exactly the class of relatively complemented distributive lattices.

The argument in this last example obviously generalizes to any locally finite, equationally complete, congruence distributive variety. Such a variety is always filtral and contains, up to isomorphism, a unique simple algebra, $\mathbf{S}$, with no proper, non-trivial subalgebras. Thus $\mathbf{S}$ is saturated, and the class of saturated algebras is $\Gamma^a(\mathbf{1}, \mathbf{S})$.

Theorems 1 and 2 together do not constitute a characterization of the saturated algebras, even in a discriminator variety. We have $\Gamma^a(\mathcal{M}^+_{\text{sat}}) \subseteq \mathscr{V}_{\text{sat}} \subseteq \Gamma^a(\mathcal{M}^+)$ for any filtral variety $\mathscr{V}$. One would hope that one of the two inclusions be an equality. However, that is not the case. Certainly the second inclusion will be proper whenever $\mathcal{M}^+$ contains a non-saturated member. The first inclusion is more work. We need a preliminary result which provides both positive and negative information.

LEMMA 4. *Let $f:\mathbf{A} \to \mathbf{B}$ be an injective epimorphism and $r:\mathbf{B} \to \mathbf{A}$ a retraction of f. Then f is an isomorphism.*

Proof. By "r is a retraction of f", we mean $r \circ f = id_A$. Let $g = f \circ r$, a map from $\mathbf{B}$ to itself. We have $g \circ f = f \circ r \circ f = f \circ id_A = id_B \circ f$. Since f is an epimorphism, this implies $g = id_B$. Thus r is the inverse of f, and f is an isomorphism. ●

THEOREM 5. *Let $\mathscr{V}$ be a filtral variety and $\mathbf{A} = \Pi\,(\mathbf{A}_i : i \in I)$, where $\mathbf{A}_i \in \mathcal{M}$, for all $i \in I$. Then $\mathbf{A} \in \mathscr{V}_{\text{sat}}$ if and only if every $\mathbf{A}_i \in \mathcal{M}_{\text{sat}}$.*

Proof. Suppose first that $\mathbf{A}_i \in \mathcal{M}_{\text{sat}}$, for all $i \in I$. Let $f:\mathbf{A} \to \mathbf{B}$ be an injective epimorphism of $\mathscr{V}$. We wish to construct a retraction r, of f, and apply the lemma. For every $i \in I$, let η_i denote the kernel of the coordinate projection $p_i:\mathbf{A} \to \mathbf{A}_i$. By the congruence extension property, the set $\{\theta \in \operatorname{Con}(\mathbf{B}) : f^{-1}(\theta) = \eta_i\}$ is non-empty and, by Zorn's lemma, it has a maximal element, ψ_i. Since η_i is completely meet-irreducible, so is ψ_i, and therefore $\mathbf{B}/\psi_i \in \mathcal{M}$. Thus the induced map f/ψ_i is an $\mathcal{M}^+$-epimorphism from $\mathbf{A}_i$ to $\mathbf{B}/\psi_i$, so by assumption, it is an isomorphism. Therefore we have a map $(f/\psi_i)^{-1} \circ p_{\psi_i}$ from $\mathbf{B}$ to $\mathbf{A}_i$. Letting r be the product of these maps for all $i \in I$, yields our desired retraction. That $r \circ f = id_A$ is a straightforward verification.

For the converse, let $\mathbf{A} \in \mathscr{V}_{\text{sat}}$, and fix $k \in I$. We show $\mathbf{A}_k \in \mathscr{M}^+_{\text{sat}}$. Let $f : \mathbf{A}_k \to \mathbf{C}$ be an injective $\mathscr{M}^+$-epimorphism. Define $\mathbf{B} = \Pi\,(\mathbf{B}_i : i \in I)$ by $\mathbf{B}_i = \mathbf{A}_i$ for $i \neq k$ and $\mathbf{B}_k = \mathbf{C}$. Let g be the natural homomorphism from $\mathbf{A}$ to $\mathbf{B}$. It suffices to show g is an epimorphism.

Let h_1, h_2 be maps from $\mathbf{B}$ to $\mathbf{D}$ such that $h_1 \circ g = h_2 \circ g$. Suppose $h_1 \neq h_2$. Then, for some $\theta \in \operatorname{Spec}(\mathbf{D})$, $p^D_\theta \circ h_1 \neq p^D_\theta \circ h_2$. Certainly $\theta \neq 1_D$. Since $\mathscr{V}$ is filtral and θ is a maximal proper congruence, $\ker\,(p^D_\theta \circ h_1 \circ g)$ is induced by an ultrafilter U, on I. Write η_U for the congruence induced by U, and p_U for the associated projection. Thus $(h_1 \circ g)^*(\theta) = (h_2 \circ g)^*(\theta) = \eta_U$. But observe that $\operatorname{Spec}(\mathbf{A})$ and $\operatorname{Spec}(\mathbf{B})$ are each in one-to-one correspondence with the set of all ultrafilters on I, together with the unit congruence, so it follows that g^* is a homeomorphism, $g^*(\eta^B_U) = \eta^A_U$ and therefore $h^*_1(\theta) = h^*_2(\theta) = \eta^B_U$. The maps are collected in Figure 1, which "almost commutes" in that $p_\theta \circ h_i = h_i/\eta_U \circ p_U$ for $i = 1, 2$. By our assumption, $h_1/\eta_U \neq h_2/\eta_U$, but $h_1/\eta_U \circ g/\eta_U = h_2/\eta_U \circ g/\eta_U$, which means that g/η_U is not an epimorphism.

$$\begin{tikzcd}
\mathbf{A} \arrow[r, "g"] \arrow[d, "p_U"'] & \mathbf{B} \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] \arrow[d, "p_U"'] & \mathbf{D} \arrow[d, "p_\theta"] \\
\mathbf{A}/\eta_U \arrow[r, "g/\eta_U"] & \mathbf{B}/\eta_U \arrow[r, shift left, "h_1/\eta_U"] \arrow[r, shift right, "h_2/\eta_U"'] & \mathbf{D}/\theta
\end{tikzcd}$$

Figure 1

Now suppose U is not the principal ultrafilter generated by $\{k\}$. Then g/η_U is surjective. For if $b \in B$, choose any $a \in A$ such that $a_i = b_i$ for every $i \neq k$. Then $g/\eta_U(a/\eta_U) = p_U \circ g(a) = g(a)/\eta_U = b/\eta_U$ since $\{i \in I : g(a)_i \neq b_i\} \subseteq \{k\} \notin U$. But then g/η_U is an epimorphism, which is a contradiction.

If, on the other hand, U is the principal ultrafilter generated by $\{k\}$, then p_U is the k-th projection congruence, $\mathbf{A}/\eta_U = \mathbf{A}_k$, $\mathbf{B}/\eta_U = \mathbf{B}_k$ and $g/\eta_U = f$. But by assumption, f is an epimorphism, so we again have a contradiction. This proves the assertion. ●

It might seem feasible to extend the above argument from direct products to boolean products in order to show $\mathscr{V}_{\text{sat}} \subseteq \Gamma^a(\mathscr{M}^+_{\text{sat}})$. If $\mathbf{A}$ is saturated and $\mathbf{A}$ is a boolean product of $(\mathbf{A}_i : i \in I)$, let $\mathbf{B}$ be a boolean product (over the same boolean space I) of $(\mathbf{B}_i : i \in I)$, in which $f_k : \mathbf{A}_k \to \mathbf{B}_k$ is an epimorphism for some $k \in I$, and the identity map for $i \neq k$. Then we could proceed as above to prove that the induced embedding of $\mathbf{A}$ into $\mathbf{B}$ is an epimorphism. The problem is that, unlike direct products, one can not simply assert the existence of such a $\mathbf{B}$.

We now present examples to show that, in fact, $\mathscr{V}_{\text{sat}} \nsubseteq \Gamma^a(\mathscr{M}^+_{\text{sat}})$, even for a discriminator variety $\mathscr{V}$. In the first example, $\mathscr{V}$ is a variety of finite type containing simple algebras of arbitrarily large cardinality. In the second. $\mathscr{V}$ has infinite type, and is residually less than 4. However we leave open the most interesting case; *if $\mathscr{V}$ is a finitely generated filtral (or discriminator) variety, is $\mathscr{V}_{\text{sat}} \subseteq \Gamma^a(\mathscr{M}^+_{\text{sat}})$?*

EXAMPLE 4. Fix a language containing two unary operation symbols, $\mathbf{f}$ and $\mathbf{g}$, and a quatenary operation symbol, $\mathbf{n}$. Let P denote the set of positive primes, and Z the set of integers. For every $p \in P$, define an algebra $\mathbf{A}_p$ in this language as follows.

$$A_p = \{0, 1, 2, \dots, p-1\},$$

$$\mathbf{f}^{\mathbf{A}}(m) = m + 1 (\operatorname{mod} p), \text{ and } \mathbf{g}^{\mathbf{A}}(m) = m - 1 \ (\operatorname{mod} p).$$

For every cardinal κ, define $\mathbf{B}_\kappa$ by

$$B_\kappa = \kappa \times Z,\ \mathbf{f}^{\mathbf{B}}(\lambda, m) = (\lambda, m+1),\ \mathbf{g}^{\mathbf{B}}(\lambda, m) = (\lambda, m-1).$$

Finally, define the algebra $\mathbf{C}_\kappa$ to be $\mathbf{B}_k \cup \{e\}$, where $e \notin B_\kappa$ and $\mathbf{f}^{\mathbf{C}}(e) = \mathbf{g}^{\mathbf{C}}(e) = e$. On every one of these algebras, $\mathbf{n}$ is to be intepreted as the normal transform.

Let $\mathscr{V}$ be the variety generated by $\mathscr{K} = \{\mathbf{C}_1\} \cup \{\mathbf{A}_p : p \in P\}$. $\mathscr{V}$ is a discriminator variety and therefore $\mathscr{M} = ISP_U(\mathscr{K})$. We claim that $\mathscr{M}$ consists of precisely the algebras described above. First of all, $P_U(\mathscr{K}) = IP_U(\mathbf{C}_1) \cup IP_U\{\mathbf{A}_p : p \in P\}$. Now, $\mathbf{C}_1$ satisfies the sentences: $(\exists!\, x)(\mathbf{f}(x) = x)$ and, for every positive integer t, $(\forall y)(y \neq \mathbf{f}(y) \rightarrow y \neq \mathbf{f}^t(y))$, as well as sentences asserting that $\mathbf{f}$ and $\mathbf{g}$ are inverse permutations of the universe and that $\mathbf{n}$ is the normal transform. (Here, $\mathbf{f}^t$ is shorthand for the composition of $\mathbf{f}$ with itself t times). Therefore, every member of $P_U(\mathbf{C}_1)$ satisfies these sentences. Thus these algebras all contain a unique idempotent as well as some number of orbits (under the permutation $\mathbf{f}$) that each resemble Z with successor and predecessor. In other words, every member of $P_U(\mathbf{C}_1)$ is isomorphic to $\mathbf{C}_\kappa$, some cardinal κ. By taking sufficiently large ultraproducts, we can get κ arbitarily large. Furthermore, every subalgebra of $\mathbf{C}_\kappa$ is isomorphic to $\mathbf{B}_\lambda$ or $\mathbf{C}_\lambda$ for $\lambda \leq \kappa$.

Now consider an ultraproduct $\mathbf{D} = \Pi(\mathbf{D}_i : i \in I)/\eta_U$, where, for every $i \in I$, $\mathbf{D}_i = \mathbf{A}_p$, some $p \in P$, and U is an ultrafilter on I. Suppose that for some $p \in P$, the set $I_p = \{i \in I : \mathbf{D}_i = \mathbf{A}_p\}$ is in U. Then, since $\mathbf{A}_p$ is finite, it follows that $\mathbf{D} = \mathbf{A}_p$. On the other hand, suppose $I_p \notin U$, for all $p \in P$. Observe that for every positive integer t, $\mathbf{A}_p \models (\exists y)(y = \mathbf{f}^t(y))$ if and only if p divides t. Since any fixed t

boolean product of $(\mathbf{A}/\theta : \theta \in \mathrm{Spec}_{\mathscr{K}}(\mathbf{A}))$ *and for every* $\theta \in \mathrm{Spec}_{\mathscr{K}}(\mathbf{A})$, $\mathbf{A}/\theta \in \mathscr{K}_{\mathrm{sat}}$. *Then* $\mathbf{A} \in \mathscr{V}_{\mathrm{sat}}$.

2. The right-to-left direction of Theorem 5 holds in any variety with the congruence extension property, with $\mathscr{M}^+$ replaced by $\mathscr{K}$ as in Remark 1.

3. For a class $\mathscr{K}$, let $\mathscr{K}_a$ denote the class of algebraically closed members of $\mathscr{K}$. In [K] Krauss proves a theorem quite like our Corollary 3: *If* $\mathscr{V}$ *is a filtral variety, then* $\Gamma^a(\mathscr{M}_a^+) \subseteq \mathscr{V}_a \subseteq \Gamma^a(\mathscr{M}^+)$. He asks the obvious question: Is $\mathscr{V}_a = \Gamma^a(\mathscr{M}_a^+)$? Example 4 above, used to answer our analogous question (in the negative) provides the same service here. We sketch the argument.

First, one easily checks that $P(\mathscr{M}_a^+) \subseteq \mathscr{V}_a$ for any filtral variety $\mathscr{V}$. Now, for every prime p, $\mathbf{A}_p$ is a maximal member of $\mathscr{M}^+$, thus is $\mathscr{M}^+$-algebraically closed. Therefore $\mathbf{E} = \Pi(\mathbf{A}_p : p \in P) \in \mathscr{V}_a$. But $\mathbf{E} \notin \Gamma^a(\mathscr{M}_a^+)$ since $\mathbf{E}$ has some $\mathbf{B}_\kappa$ as a homomorphic image, and $\mathbf{B}_\kappa \notin \mathscr{M}_a^+$. To see this, observe that $\mathbf{B}_\kappa < \mathbf{C}_\kappa$ and $\mathbf{C}_\kappa \models (\exists x)(\mathbf{f}(x) = x)$ whereas $\mathbf{B}_\kappa$ does not.

I would like to thank Roger Maddux for numerous helpful discussions during the preparation of this paper.

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