CATEGORICAL EQUIVALENCE AND CENTRAL RELATIONS

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A finite algebra is called *preprimal* if its clone of term operations is a coatom in the lattice of clones. These algebras are of interest both as a generalization of primal algebras, and as a toehold in the difficult analysis of the lattice of clones on a finite set.

Two algebras $\mathbf{A}$ and $\mathbf{B}$ are called *categorically equivalent* if the varieties $\mathsf{V}(\mathbf{A})$ and $\mathsf{V}(\mathbf{B})$ are equivalent as categories via a functor mapping $\mathbf{A}$ to $\mathbf{B}$. We write $\mathbf{A} \equiv_{\mathrm{c}} \mathbf{B}$ to indicate this relationship. It was shown in [5] that if $\mathbf{A}$ is primal, then $\mathbf{A} \equiv_{\mathrm{c}} \mathbf{B}$ if and only if $\mathbf{B}$ is primal. This was extended to preprimal algebras in [3] and [4]. Unfortunately, the treatment of one case—that of central relations—is incorrect in the second paper and is difficult to follow in the first. The purpose of this note is to provide a clear and straightforward argument for this one case.

Let e be an equivalence relation on $\{1, 2, \dots, h\}$, for some positive integer h, and let A be a set. We let

$$\delta_e = \big\{(x_1, x_2, \dots, x_h) \in A^h : (i, j) \in e \implies x_i = x_j\big\}.$$

Relations of the form δ_e are called *generalized diagonals,* and are invariant under every operation on A. We will call e nontrivial if $\delta_e \neq A^h$.

Definition. Let $\mathbf{A}$ be a finite set, h a positive integer, and $\rho \subsetneq A^h$. Then ρ is an *h-ary central relation* on A if

- for every nontrivial e, $\delta_e \subseteq \rho$;
- for every permutation σ of $\{1, 2, \dots, h\}$, $(x_1, x_2, \dots, x_h) \in \rho \implies (x_{\sigma 1}, x_{\sigma 2}, \dots, x_{\sigma h}) \in \rho$;
- there is a nonempty subset $Z(\rho)$ of A such that $Z(\rho) \times A^{h-1} \subseteq \rho$.

The set $Z(\rho)$ is called the *center* of ρ.

Let Θ be a set of relations on a set A. By $\mathcal{P}(\Theta)$ we mean the clone of all operations preserving every member of Θ. Let us call a finite algebra $\mathbf{A}$ of *h-ary central type* if the clone of term operations on $\mathbf{A}$ is equal to $\mathcal{P}(\rho)$ for some h-ary central relation ρ on A. Unary central type is a special case of subalgebra-primality, which is considered in [2]. For the remainder of this paper we restrict to the case that $h > 1$.

Let h be an integer greater than 1, and $B_h = \{0, 1, 2, \dots, h\}$. We define

$$\nu_h = \big\{\, (x_1, x_2, \dots, x_h) \in B_h^h : \{x_1, \dots, x_h\} \neq \{1, 2, \dots, h\} \,\big\}.$$

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(Equivalently, ν_h is the set of those h-tuples containing at least one component equal to 0, or at least one pair of equal components.) Note that ν_h is the unique central relation on B_h such that $Z(\nu_h) = \{0\}$. Finally, let $\mathbf{B}_h = \langle B_h, \mathcal{P}(\nu_h) \rangle$.

Theorem. *Let h be an integer greater than 1 and let $\mathbf{A}$ be of h-ary central type. Then $\mathbf{A} \equiv_{\mathrm{c}} \mathbf{B}_h$.*

Proof. Let ρ be the h-ary central relation on A guarenteed by the hypothesis. By McKenzie's theorem [6, Corollary 6.1], it suffices to find an invertible idempotent term s on $\mathbf{A}$ such that $\mathbf{A}(s)$ is weakly isomorphic to $\mathbf{B}_h$. By Theorem 2.1 (and the remarks following the proof) of [2] and Lemma 2.4 of [4], this is equivalent to finding an operation $s \in \mathcal{P}(\rho)$ such that

(1) For all $x \in A$, $s(s(x)) = s(x)$;
(2) $\mathcal{P}(\rho)$ contains an $(h+1)$-ary near unanimity term;
(3) For every pair θ, ψ of distinct subalgebras of $\mathbf{A}^h$, $s(\theta) \neq s(\psi)$;
(4) The relational structures $\langle s(A), s(\rho) \rangle$ and $\langle B_h, \nu_h \rangle$ are isomorphic.

In 3 and 4, by $s(\theta)$ we mean $\left\{ (s(x_1), \dots, s(x_h)) : (x_1, x_2, \dots, x_h) \in \theta \right\}$.

Since $\rho \neq A^h$, there is $\mathbf{a} = (a_1, a_2, \dots, a_h) \in A^h - \rho$. From the definition of central relation, for every $1 \leq i < j \leq n$ we have $a_i \neq a_j$ and $a_i \notin Z(\rho)$. Since the center is nonempty, we fix an element a_0 of $Z(\rho)$.

Define the unary operation s on A by

$$s(x) = \begin{cases} x & \text{if } x \in \{a_0, a_1, \dots, a_h\} \\ a_0 & \text{otherwise.} \end{cases}$$

Our first task is to prove that s preserves ρ. So let $\mathbf{x} = (x_1, \dots, x_h) \in \rho$ and $\mathbf{y} = (y_1, \dots, y_h) = s(\mathbf{x})$. If some pair of components of $\mathbf{x}$ are equal, then the corresponding pair of components of $\mathbf{y}$ are equal, thus $\mathbf{y} \in \rho$. So suppose that the components of $\mathbf{x}$ are pairwise distinct. Since no permutation of $\mathbf{a}$ is in ρ, there must be an $i \leq h$ such that $y_i = s(x_i) = a_0$, so $\mathbf{y} \in \rho$. We conclude that $s \in \mathcal{P}(\rho)$.

It is obvious from its definition that s is idempotent (i.e., condition 1 holds) and $s(A) = \{a_0, a_1, \dots, a_h\}$. Furthermore, the mapping $a_i \mapsto i$, for $i = 0, 1, 2, \dots, h$ satisfies condition 4. It is well-known that every central relation admits a near-unanimity term. In fact, we can obtain such a term by defining $m(x_0, x_1, \dots, x_h)$ to be a_0 whenever the "near-unanimity" conditions do not apply.

We now consider condition 3. For $\mathbf{x} \in A^h$, let $e(\mathbf{x}) = \{ (i,j) : x_i = x_j \}$. We require the following Lemma.

Lemma. *Let $\mathbf{x} \in A^h$, and let θ be the subalgebra of $\mathbf{A}^h$ generated by $\mathbf{x}$. If $e(\mathbf{x})$ is nontrivial, then $\theta = \delta_{e(\mathbf{x})}$. If $e(\mathbf{x})$ is trivial, then $\theta = \rho$ if $\mathbf{x} \in \rho$, else, $\theta = A^h$.*

Proof of Lemma. Let $\mathbf{y}$ be any element of the subalgebra that, according to the statement of the Lemma, is supposed to be equal to θ. Define a unary operation f by $f(x_i) = y_i$, for $i = 1, 2, \dots, h$, and $f(w) = a_0$ otherwise. Since $e(\mathbf{y}) \supseteq e(\mathbf{x})$, f is well-defined. It suffices to prove that $f \in \mathcal{P}(\rho)$.

So let $\mathbf{z} \in \rho$. If $\mathbf{z}$ has a pair equal components, then so does $f(\mathbf{z})$, so $f(\mathbf{z}) \in \rho$. Thus we can assume that the components of $\mathbf{z}$ are pairwise distinct. If, for some $i \leq h$, $z_i \notin \{x_1, \dots, x_h\}$, then $f(z_i) = a_0 \in Z(\rho)$, so again, $f(\mathbf{z}) \in \rho$. The only remaining possibility is that $\mathbf{z}$ is a permutation of $\mathbf{x}$. In that case, $\mathbf{x} \in \rho$, so $\mathbf{y} \in \rho$. Since $f(\mathbf{z})$ is a permutation of $\mathbf{y}$, we conclude that $f(\mathbf{z}) \in \rho$. □

Now we verify condition 3. Let θ, ψ be subalgebras, and assume that $\theta \nsubseteq \psi$. Therefore, there is a join-irreducible subalgebra μ such that $\mu \subseteq \theta$ and $\mu \nsubseteq \psi$. Since every join-irreducible subalgebra is 1-generated, it follows from the Lemma that either μ is a generalized diagonal, or $\mu = \rho$. For each of the possibilities, we show, via the Lemma, that there is a generator $\mathbf{x}$ of μ with $\mathbf{x} \in s(A)^h$. Then $\mathbf{x} \in s(\theta) - s(\psi)$ as desired.

If $\mu = A^h$, we take $\mathbf{x} = (a_1, a_2, \dots, a_h)$. If $\mu = \rho$, take $\mathbf{x} = (a_0, a_2, \dots, a_h)$. Finally, suppose that $\mu = \delta_e$ with e nontrivial. Choose any $\mathbf{x} \in \{a_1, \dots, a_h\}^h$ with $e(\mathbf{x}) = e$. □

Remarks. It follows from the Lemma that if $\mathbf{A}$ is of h-ary central type, then the only join-irreducible members of $\mathrm{Sub}(\mathbf{A}^h)$ are generalized diagonal relations and, possibly, the central relation ρ. In fact, if $h = 2$, then ρ is indeed join-irreducible. However, it is easy to show that if $h > 2$ then there are distinct nontrivial equivalence relations e and e', such that $\delta_e \vee \delta_{e'} = \rho$.

The notion of a c-minimal algebra was introduced in [1]. A finite c-minimal algebra, if it exists, is unique in its categorical equivalence class. It follows from the proof of the Theorem that every $\mathbf{B}_h$ is c-minimal. As a corollary we obtain: If $h \neq k$ then $\mathbf{B}_h \not\equiv_{\mathrm{c}} \mathbf{B}_k$. If $\mathbf{A}$ is h-ary central and $\mathbf{A}'$ is k-ary central, then $\mathbf{A} \not\equiv_{\mathrm{c}} \mathbf{A}'$. No algebra can be both h-central and k-central.

References

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