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How to cancel a linearly ordered exponent

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In this paper, X^Y denotes the set of monotone functions from the partially ordered set Y into the partially ordered set X, ordered by everywhere inclusion. Several cancellation and refinement theorems for this operation on po-sets were announced in [2], and will be proved in [3]. The purpose of this short paper is to establish the following theorem. The proof is a good example of the circle of ideas elaborated more fully in [3].

Theorem. *If L is linearly ordered and $A^L \cong B^L$, then $A \cong B$.*

The first result along the lines of this theorem was obtained by Jónsson in spring 1977. He obtained the conclusion if $L=2$ and A is either atomic or else is a lattice. Jónsson's ideas for the proof turned out to have quite general applicability. In the span of a year, Jónsson's argument was improved in three stages, first by McKenzie, then by Bergman, and finally by Nagy, to produce the theorem stated above.

We first handle the case $L=2$, as it seems easier to grasp the general argument after looking at the special case.

Suppose that $\varphi: A^2 \cong B^2$. We shall denote elements of A^2 as ordered pairs $\langle x, y\rangle$, where $x \leqq y$ in A. What we shall show is that for any $a \in A$ there are unique $\bar{b}_i \in B$, $a_i \in A$ $(i=0,1)$ and $\bar{b} \in B$ such that:

$$\varphi(\langle a, a\rangle) = \langle \bar{b}_0, \bar{b}_1\rangle,$$
$$\varphi(\langle a_i, a_i\rangle) = \langle \bar{b}_i, \bar{b}_i\rangle, \quad i = 0,1,$$
$$\varphi(\langle a_0, a_1\rangle) = \langle \bar{b}, \bar{b}\rangle.$$

This gives a mapping of A into B: $\hat{\varphi}(a)=\bar{b}$ when the above equations hold. Now the same fact holds for φ^{-1}. And from the above equations it clearly follows that $\widehat{(\varphi^{-1})}=(\hat{\varphi})^{-1}$. Thus $\hat{\varphi}: A \cong B$.

Before beginning the proof (for $L=2$) let us remark that although our structures are not assumed to be lattices, it is convenient to use the notations '$\wedge$' and '$\vee$'.

Here $a \wedge b$ will denote the greatest lower bound of a and b, and we only use it in situations where $a \wedge b$ obviously exists. We remark that if $f \wedge g$ exists in A^L then $\varphi(f) \wedge \varphi(g)$ exists in B^L and equals $\varphi(f \wedge g)$.

Assume now that $\varphi(\langle a, a\rangle)=\langle \bar b_0, \bar b_1\rangle$ and that $\varphi(\langle x, y\rangle)=\langle \bar b_1, \bar b_1\rangle$. We claim that $x=y$. Suppose that, to the contrary, $x<y$. Then we have $\langle x, y\rangle<\langle y, y\rangle$ and so

$$\varphi(\langle y, y\rangle) = \langle \bar p, \bar q\rangle, \quad \bar b_1 \leqq \bar p \leqq \bar q, \quad \bar b_1 < \bar q.$$

Then $\varphi^{-1}(\langle \bar b_1, \bar q\rangle)$ must be of the form $\langle x_1, y\rangle$ with $x<x_1 \leqq y$. We write $\varphi^{-1}(\langle \bar b_0, \bar q\rangle)=\langle u, v\rangle$.

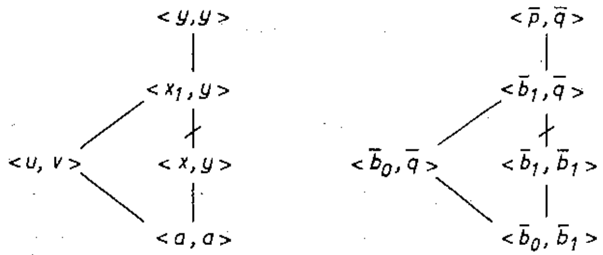

Figure 1

Now we have:

$$\langle \bar b_0, \bar q\rangle \wedge \langle \bar b_1, \bar b_1\rangle = \langle \bar b_0, \bar b_1\rangle,$$

$$\langle \bar b_0, \bar q\rangle \vee \langle \bar b_1, \bar b_1\rangle = \langle \bar b_1, \bar q\rangle \quad \text{in } B^2,$$

hence

$$\langle u, v\rangle \wedge \langle x, y\rangle = \langle a, a\rangle.$$

But $a \leqq u \leqq v \leqq y$, $a \leqq x$, so $\langle a, v\rangle \leqq \langle u, v\rangle \wedge \langle x, y\rangle$.

Thus $v=a$, and it follows that $\langle u, v\rangle=\langle a, a\rangle$. This is impossible because $\langle \bar b_0, \bar q\rangle > \langle \bar b_0, \bar b_1\rangle$.

By reasoning exactly dual to the above, it is shown that $\varphi^{-1}(\langle \bar b_0, \bar b_0\rangle)$ is constant. Let $\varphi(\langle a_i, a_i\rangle)=\langle \bar b_i, \bar b_i\rangle$ for $i=0, 1$ and let us now consider $\varphi(\langle a_0, a_1\rangle)=\langle \bar b, \bar c\rangle$, to show that $\bar b=\bar c$. All functions mentioned below in A^2 lie in between $\langle a_0, a_0\rangle$ and $\langle a_1, a_1\rangle$; in B^2 they lie in between $\langle \bar b_0, \bar b_0\rangle$ and $\langle \bar b_1, \bar b_1\rangle$. We have, clearly, $\varphi^{-1}(\langle \bar b, \bar b\rangle)=\langle a_0, y\rangle$ where $a_0 \leqq y \leqq a_1$. Moreover, $\langle \bar b, \bar b\rangle \vee \langle \bar b_0, \bar b_1\rangle=\langle \bar b, \bar c\rangle \vee \langle \bar b_0, \bar b_1\rangle= =\langle \bar b, \bar b_1\rangle$, hence $\langle a_0, y\rangle \vee \langle a, a\rangle=\langle a_0, a_1\rangle \vee \langle a, a\rangle=\langle a, a_1\rangle$. This requires $y \vee a=a_1$ in A. Thus $\langle a, a\rangle \vee \langle y, y\rangle=\langle a_1, a_1\rangle$, and $\langle \bar b_0, \bar b_1\rangle \vee \varphi(\langle y, y\rangle)=\langle \bar b_1, \bar b_1\rangle$. But if $\varphi(\langle y, y\rangle)=\langle \bar p, \bar q\rangle$, since $\bar b_0 \leqq \bar p \leqq \bar q \leqq \bar b_1$, then $\langle \bar b_0, \bar b_1\rangle \vee \varphi\langle y, y\rangle=\langle \bar p, \bar b_1\rangle$. We conclude that $\varphi\langle y, y\rangle=\langle \bar b_1, \bar b_1\rangle$, hence $y=a_1$, so $\langle \bar b, \bar b\rangle=\langle \bar b, \bar c\rangle$ as desired.

We now return to the general case. The proof is only slightly more complicated and requires a few supplementary notions. Suppose from now on that A and B are arbitrary po-sets and L is a linear order. Let $\varphi\colon A^L \cong B^L$. We denote elements of L by $i, j, k, \ldots$; elements of A by $a, b, c, \ldots$; and elements of A^L by $\beta, \gamma, \delta, \ldots$. Elements of B and B^L will be denoted by $\bar a, \bar b, \bar c$ and $\bar\beta, \bar\gamma, \bar\delta$ respectively. In

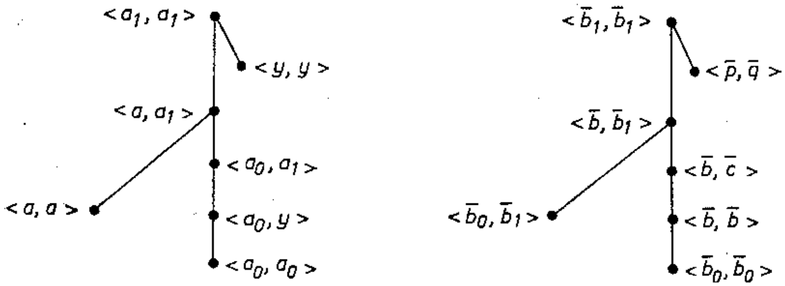

Figure 2

addition, the element of A^L which takes the value 'a' for $i<j$ and 'b' otherwise may be denoted by $\langle a[i<j], b\rangle$ and the constant function 'a' by $\langle a\rangle$.

Definition. 1) $\Delta(\varphi)=\{\beta\in A^L\colon \varphi(\beta)$ *is constant*$\}$.
2) $R(\varphi)=\{a\in A\colon \langle a\rangle\in\Delta(\varphi)\}$.
3) $a\leqq_\varphi b$ *iff* $a, b\in R(\varphi)$, $a\leqq b$ *and* $\{\beta\in A^L\colon \beta(L)=\{a, b\}\}\subseteq\Delta(\varphi)$.
4) β^i *is defined by* $\beta^i(j)=\beta(\min(i, j))$,
β_i *is defined by* $\beta_i(j)=\beta(\max(i, j))$.

Lemma 1. *If* $\beta\in\Delta(\varphi)$ *and* $i\in L$ *then* $\beta^i\in\Delta(\varphi)$ *and* $\beta_i\in\Delta(\varphi)$.

Proof. Because of duality it is enough to prove the statement $\beta^i\in\Delta(\varphi)$. Let $\varphi(\beta)=\langle\bar{x}\rangle$ and $\varphi(\beta^i)=\bar{\gamma}$. We need to show $\bar{\gamma}$ constant. If not there is $j\in L$ such that $\bar{\gamma}^j\neq\bar{\gamma}$. Let $\bar{\delta}=\bar{\gamma}^j$. Then $\bar{\delta}<\bar{\gamma}\leqq\langle\bar{x}\rangle$, hence $\varphi^{-1}(\bar{\delta})=\delta<\beta^i\leqq\beta$.

Now let $\lambda=\langle\delta[k\leqq i], \beta\rangle$. Then $\delta\leqq\lambda\leqq\beta$. Thus $\bar{\lambda}=\varphi(\lambda)\geqq\bar{\delta}$, hence $\bar{\lambda}^j\geqq\bar{\delta}^j=\bar{\gamma}^j$. Also $\lambda\vee\beta^i$ exists and $\lambda\vee\beta^i=\beta$, so $\bar{\lambda}\vee\bar{\gamma}=\langle\bar{x}\rangle$. Thus $\bar{\lambda}\geqq\bar{\lambda}^j\vee\bar{\gamma}^j=(\bar{\lambda}\vee\bar{\gamma})^j=\langle\bar{x}\rangle^j=\langle\bar{x}\rangle$, so $\bar{\lambda}=\langle\bar{x}\rangle$, and applying φ^{-1}, $\lambda=\beta$. But $\lambda^i=\delta^i<\beta^i$, contradiction.

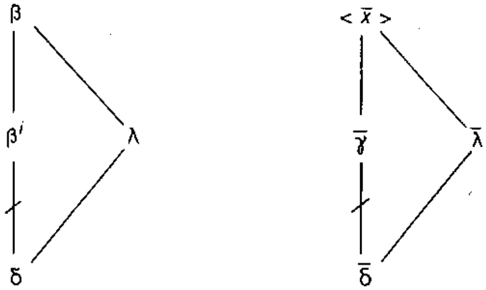

Figure 3

A quick comparison will reveal that this argument is essentially no different from the one for $L=2$. □

Lemma 2. *Suppose that* $a, b\in R(\varphi)$, $a\leqq b$, $\varphi(\langle a\rangle)=\langle\bar{a}\rangle$, $\varphi(\langle b\rangle)=\langle\bar{b}\rangle$, U, U_1, U_2 *are initial segments of* L, V, V_1, V_2 *are final segments of* L, $\bar{\beta}\in B^L$, $\bar{\beta}(U)=\{\bar{a}\}$,

$\bar{\beta}(V)=\{\bar{b}\}$, $\bar{\beta}=\varphi(\beta)$, $|\beta(U_1)|\leqq 1$, $|\beta(V_1)|\leqq 1$, $U_2\cup V_2=L$, $\gamma(U_2)=\{a\}$, $\gamma(V_2)=\{b\}$, $\bar{\gamma}=\varphi(\gamma)$. *Then* $V_2\neq V_1$, $V_2\subset V_1$ *implies that if* $\bar{\delta}\geqq\bar{\gamma}$, $\bar{\delta}{\restriction}L-U=\bar{\gamma}{\restriction}L-U$ *then* $\bar{\delta}=\bar{\gamma}$; *equivalently,* $\bar{\gamma}$ *is constant on* U *and this constant is a maximal lower bound for* $\bar{\gamma}(L-U)$.

Dually $U_2\neq U_1$, $U_2\subset U_1$ *implies that if* $\bar{\delta}\leqq\bar{\gamma}$, $\bar{\delta}{\restriction}L-V=\bar{\gamma}{\restriction}L-V$ *then* $\bar{\delta}=\bar{\gamma}$; *equivalently,* $\bar{\gamma}$ *is constant on* V *and this constant is a minimal upper bound for* $\bar{\gamma}(L-V)$.

Proof. We prove the first of the two dual statements. Suppose that the implication does not hold, i.e. $\bar{\delta}>\bar{\gamma}$, $\bar{\delta}{\restriction}L-U=\bar{\gamma}{\restriction}L-U$ for some $\bar{\delta}=\varphi(\delta)$. As $\gamma\wedge\beta$ exists, $\bar{\gamma}\wedge\bar{\beta}$ and so $\bar{\delta}\wedge\bar{\beta}=\bar{\gamma}\wedge\bar{\beta}$ also exist, which implies $\delta\wedge\beta=\gamma\wedge\beta$. As $\delta>\gamma$, for some $i\in U_2\wedge V_1$ $\delta(i)>a$. Let $\lambda=\delta^i>\langle a\rangle$.

Since $\delta\wedge\beta=\gamma\wedge\beta$, $(\delta\wedge\beta)^i=(\gamma\wedge\beta)^i=\langle a\rangle$, so $\langle a\rangle=(\delta\wedge\beta)^i=\delta^i\wedge\beta^i=\lambda\wedge\beta$, and applying φ, $\bar{\lambda}\wedge\bar{\beta}=\langle\bar{a}\rangle$, where $\bar{\lambda}=\varphi(\lambda)$. Thus for $j\in V$ $\bar{\lambda}(j)=(\bar{\beta}\wedge\bar{\lambda})(j)=\bar{a}$, hence $\bar{\lambda}=\langle\bar{a}\rangle$, $\lambda=\langle a\rangle$: contradiction. □

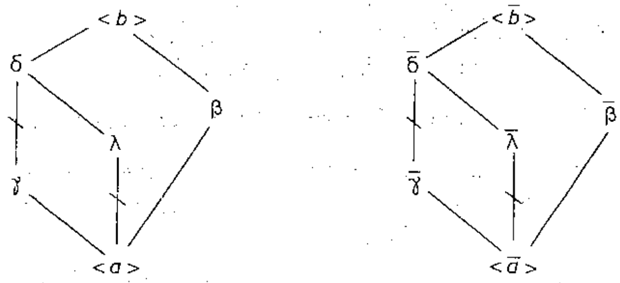

Figure 4

Lemma 3. *For any* $a,b\in R(\varphi)$ *with* $a\leqq b$ *the following are equivalent (where* $\varphi(\langle a\rangle)=\langle\bar{a}\rangle$, $\varphi(\langle b\rangle)=\langle\bar{b}\rangle$*):*

i) $a\leqq_\varphi b$.

ii) $\bar{a}\leqq_{\varphi^{-1}}\bar{b}$.

iii) *There is some* $\beta\in\Delta(\varphi)$ *with* $\beta(L)=\{a,b\}$.

iv) *There is some* $\bar{\beta}\in\Delta(\varphi^{-1})$ *with* $\bar{\beta}(L)=\{\bar{a},\bar{b}\}$.

Proof. We can clearly assume that L has more than one element. Then (i)$\Rightarrow\langle a[i\leqq j], b\rangle\in\Delta\Rightarrow$(iii) where $j<k$ in L. By symmetry, it suffices now to prove (iv) implies (i). So suppose $\bar{\beta}$ is as in (iv), $\varphi^{-1}(\bar{\beta})=\beta=\langle c\rangle$. Let γ be any element of A^L with $\gamma(L)=\{a,b\}$. We show $\varphi(\gamma)=\bar{\gamma}$ is constant. $U=\bar{\beta}^{-1}(a)$, $U_2=\gamma^{-1}(a)$ are initial segments of L and $V=\bar{\beta}^{-1}(\beta)$, $V_2=\gamma^{-1}(b)$ are final segments of L. Both parts of Lemma 2 can be applied, the first with $U_1=\emptyset$, $V_1=L$, the second with $U_1=L$, $V_1=\emptyset$. Therefore $\bar{\gamma}$ is constant on U and on $V=L-U$, and the two constants are equal. □

Lemma 4. *i) If* $a_0 \leqq a_1 \leqq a_2$ $(a_i \in R(\varphi))$, *then* $a_0 \leqq_\varphi a_2$ *iff* $a_0 \leqq_\varphi a_1$ *and* $a_1 \leqq_\varphi a_2$.
ii) $\leqq_\varphi$ *is a subordering of* $\leqq$ *on* $R(\varphi)$.
iii) $\mathring{\varphi}: \langle R(\varphi), \leqq_\varphi\rangle \cong \langle R(\varphi^{-1}), \leqq_{\varphi^{-1}}\rangle$ *where* $\varphi(\langle a\rangle) = \langle\mathring{\varphi}(a)\rangle$.

Proof. ii) follows from i), and iii) follows from Lemma 3. To prove i) let U be a nontrivial initial segment of L, $\beta_{02} = \langle a_0[i \in U], a_2\rangle$, $\beta_{01} = \langle a_0[i \in U], a_1\rangle$ and $\beta_{12} = \langle a_1[i \in U], a_2\rangle$. Suppose first that $a_0 \leqq_\varphi a_2$. Then:

$$\beta_{01} = \langle a_1\rangle \wedge \beta_{02}$$

$$\beta_{12} = \langle a_1\rangle \vee \beta_{02}.$$

By hypothesis, the four functions on the right side of the equal signs are members of $\Delta(\varphi)$. Since $\Delta(\varphi)$ is closed under meets and joins we conclude $a_0 \leqq_\varphi a_1$ and $a_1 \leqq_\varphi a_2$.

To prove the converse, let $\bar{b}_i = \mathring{\varphi}(a_i)$ $(i = 0, 1, 2)$. We have to show that $\varphi(\beta_{02}) = \bar{\beta}_{02}$ is constant, i.e. $\bar{\beta}^i_{02} = \bar{\beta}_{02}$ for each $i \in L$. With $\tau = \varphi^{-1}(\bar{\beta}^i_{02})$ we are to prove $\tau = \beta_{02}$. Clearly $\beta_{02} \geqq \tau$. By assumption $a_0 \leqq_\varphi a_1$, so the image of β_{01} is a constant, say $\langle\bar{w}\rangle$. Since β_{02} lies above the former, we have $\bar{\beta}_{02} \geqq \langle\bar{w}\rangle$ and therefore $\bar{\beta}^i_{02} \geqq \langle\bar{w}\rangle$. Applying φ^{-1} to this $\tau \geqq \beta_{01}$. As $\tau \leqq \beta_{02}$, then $\tau(U) = \{a_0\}$ and $a_1 \leqq \tau(j) \leqq a_2$ for $j \in L - U$. Thus $\langle a_1\rangle \vee \tau = \langle a_1[j \in U], \tau\rangle$. It will be enough then to show that $\langle a_1\rangle \vee \tau \geqq \beta_{12}$ to get $\tau \geqq \beta_{02}$, or that $\langle\bar{b}_1\rangle \vee \bar{\beta}^i_{02}$ (which we have proved to exist) dominates $\varphi(\beta_{12}) = \langle\bar{z}\rangle$.

Now putting $\langle\bar{b}_1\rangle \vee \bar{\beta}^i_{02} = \bar{\sigma}$, and $\bar{\xi} = \langle\bar{\sigma}[j \leqq i], \bar{b}_2\rangle$, we have $\bar{\xi} \geqq \langle\bar{b}_1\rangle, \bar{\beta}_{02}$, hence $\bar{\xi} \geqq \langle\bar{z}\rangle$ as $\langle\bar{b}_1\rangle \vee \bar{\beta}_{02} = \langle\bar{z}\rangle$. Thus $\bar{\sigma}(j) \geqq \bar{z}$ if $j \leqq i$, implying that $\bar{\sigma} \geqq \langle\bar{z}\rangle$ as desired. □

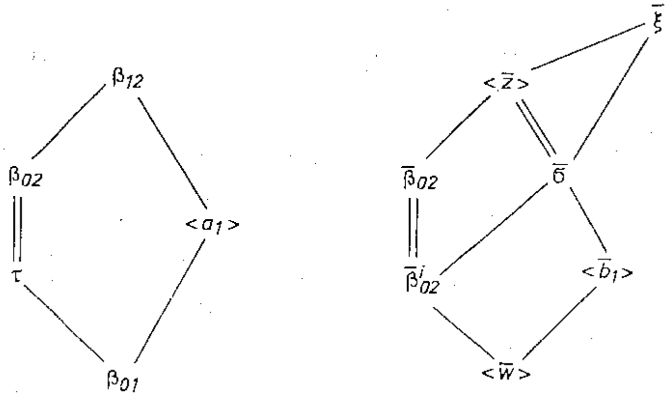

Figure 5

Proposition 5. *If* L *is linearly ordered, then* $\Delta(\varphi)$ *is the universe of* $\langle R(\varphi), \leqq_\varphi\rangle^L$.

Proof. Let $\beta \in A^L$. If β is in the universe of $\langle R(\varphi), \leqq_\varphi\rangle^L$, $\beta(j) \geqq_\varphi \beta(i)$ for $j \geqq i$ in L so $\langle\beta(i)[k < j], \beta(j)\rangle \in \Delta(\varphi)$. It is easily seen that

$$\beta_i = \bigvee\{\langle\beta(i)[k < j], \beta(j)\rangle: j \in L, j \geqq i\}.$$

Since $\Delta(\varphi)$ is closed under arbitrary joins (when they exist) we have $\beta_i \in \Delta(\varphi)$. As the same is true for meets, we have $\bigwedge\{\beta_i \colon i \in L\} = \beta \in \Delta(\varphi)$.

To prove the converse, suppose $\beta \in \Delta(\varphi)$. $\langle \beta(i) \rangle = (\beta_i)^i$ so by Lemma 1, applied twice, $\langle \beta(i) \rangle \in \Delta(\varphi)$, $\beta(i) \in R(\varphi)$. Let $i, j \in L$, $i < j$. We have to prove $\beta(i) \leqq_\varphi \beta(j)$. We may assume $\beta = (\beta_i)^j$, because $(\beta_i)^j \in \Delta(\varphi)$ by Lemma 1. If $\beta(L) = \{\beta(i), \beta(j)\}$, the statement is proven by Lemma 3. Otherwise we have a k with $i < k < j$ and $\beta(i) < \beta(k) < \beta(j)$. By Lemma 4 i) it is enough to prove $\beta(i) \leqq_\varphi \beta(k)$ and $\beta(k) \leqq_\varphi \beta(j)$. The two inequalities are symmetric, we prove only the first.

Let $\lambda = \langle \beta(i)[m<j], \beta(k) \rangle$. By Lemma 3 it suffices to show $\lambda \in \Delta(\varphi)$. Suppose it is not so. Then $\bar{\lambda} = \varphi(\lambda)$ is not a constant, so there exists an m such that $\bar{\lambda} \neq \bar{\lambda}_m$. Let $\bar{\tau} = \langle \mathring{\varphi}(\beta(i))[n<m], \mathring{\varphi}(\beta(k)) \rangle$. Apply the second statement of Lemma 2 for φ^{-1} with the following substitutions:

$$\bar{a} = \beta(i), \quad \bar{b} = \beta(k), \quad \bar{\beta} = \beta^k, \quad U = \{n \in L \colon n \leqq i\}, \quad V = \{n \in L \colon n \geqq k\},$$

$$U_1 = L, \quad V_1 = \emptyset, \quad U_2 = \{n \in L \colon n < m\}, \quad V_2 = \{n \in L \colon n \geqq m\}, \quad \gamma = \bar{\tau}.$$

Then the statement of the Lemma is: $\tau = \varphi^{-1}(\bar{\tau})$ is constant on $\{n \in L, n \geqq k\}$. Now apply the first statement of Lemma 2 with the following substitutions:

$$\bar{a} = \mathring{\varphi}(\beta(i)), \quad \bar{b} = \mathring{\varphi}(\beta(k)), \quad \bar{\beta} = \bar{\tau}, \quad U = \{n \in L \colon n < m\}, \quad V = \{n \in L \colon n \geqq m\},$$

$$U_1 = \emptyset, \quad V_1 = \{n \in L, n \geqq k\}, \quad U_2 = \{n \in L \colon n < j\}, \quad V_2 = \{n \in L, n \geqq j\}, \quad \gamma = \lambda.$$

$k \in V_1$ and $k \notin V_2$ so $V_1 \supsetneqq V_2$, and the other hypotheses also hold. We obtain $\bar{\lambda}$ is constant on $\{n \in L \colon n < m\}$ and is equal to the maximal lower bound of $\bar{\lambda}(\{n \in L \colon n \geqq m\})$, which is $\bar{\lambda}(m)$. But then $\bar{\lambda} = \bar{\lambda}_m$: contradiction. □

Proof of the Theorem. We have $\varphi \colon A^L \cong B^L$, L linearly ordered. For any $a \in A$ we consider $\bar{\beta} = \varphi(\langle a \rangle)$. By definition $\bar{\beta} \in \Delta(\varphi^{-1})$ so by Proposition 5, $\bar{\beta} \in \langle R(\varphi^{-1}), \leqq_\varphi \rangle^L$.

But by Lemma 4 (iii) we have $\mathring{\varphi} \colon \langle R(\varphi), \leqq_\varphi \rangle \cong \langle R(\varphi^{-1}), \leqq_{\varphi^{-1}} \rangle$, so again by Proposition 5:

$$\beta = \mathring{\varphi}^{-1} \circ \bar{\beta} \in \Delta(\varphi).$$

Thus $\varphi(\beta) = \langle \bar{a} \rangle$ for some $\bar{a} \in B$. Then $\hat{\varphi} \colon A \cong B$ is given by $\hat{\varphi}(a) = \bar{a}$. In symbols $\hat{\varphi} = \pi^{-1} \circ \varphi \circ \bar{\varphi}^{-1} \circ \varphi \circ \pi$ where π is the inclusion of a po-set into its "powers", via the constant functions and $\bar{\varphi}$ is the extension of $\mathring{\varphi}$ from $\langle R(\varphi), \leqq_\varphi \rangle$ to $\langle R(\varphi), \leqq_\varphi \rangle^L$. All the functions involved are clearly order preserving and the map $\pi^{-1} \circ \varphi^{-1} \circ \bar{\varphi} \circ \varphi^{-1} \circ \pi$ is easily seen to be inverse to $\hat{\varphi}$. □

Remarks. The theorem of this paper is the only cancellation result we know of that imposes no conditions on the bases (that is, on A and B). However, the structural facts about an isomorphism $\varphi \colon A^D \cong B^E$ which we have revealed in this paper in case $D = E$ is linearly ordered, are true much more generally, as will be shown in [3]. A typical result of that paper is the following: If $A^D \cong B^E$, all structures are finite, and D, E are connected, then (provided A has a least element, or D and E have least elements) there exist R, D', E' such that $A \cong R^{E'}, B \cong R^{D'}$, and $DE' \cong ED'$. Moreover, if $D \cong E$ then $A \cong B$. When D and E are directly, indecomposable and not isomorphic, the po-sets R, D', E' can be chosen to be $\langle R(\varphi), \leqq_\varphi \rangle, D, E,$ respectively.

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