

Notes on Boolean semilattices

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1. PRELIMINARIES

1. SEMILATTICES

A (meet) semilattice is an algebra $\mathbf{S} = \langle S, \cdot \rangle$ satisfying

$$x \cdot x = x, \quad x \cdot y = y \cdot x, \quad x \cdot (y \cdot z) = (x \cdot y) \cdot z.$$

As usual, we define an ordering on S by $x \preceq y \iff x \cdot y = x$. We will also think of the semilattice \mathbf{S} as a ternary relational structure $\langle S, R_S \rangle$, in which $R_S = \{ (x, y, z) \in S^3 : x \cdot y = z \}$. When absolutely necessary, we can write \mathbf{S}^\square for this relational structure. We will reserve the letter R for these ternary relations. The least and greatest elements (if any) of a semilattice will be denoted \perp and \top .

Finally, we define \mathbf{S} to be the class of all semilattices. Generally, we think of this as a class of relational structures.

2. THE COMPLEX ALGEBRA OF A SEMILATTICE

Let \mathbf{S} be a semilattice. The *complex algebra* of \mathbf{S} is the algebra

$$\mathbf{S}^+ = \langle \mathcal{P}(S), \cup, \cap, ', \emptyset, S, \cdot \rangle,$$

where, for $X, Y \subset S$, we define $X \cdot Y = \{ x \cdot y : x \in X, y \in Y \}$. \mathbf{S}^+ is, of course, a Boolean algebra with a normal binary operator—a structure we shall call a Boolean groupoid. We denote the natural ordering on any *ba* by ' \leq '.

We define \mathbf{S}^+ to be the class of all algebras (isomorphic to??) \mathbf{S}^+ for $\mathbf{S} \in \mathbf{S}$, and $\mathbf{BSI} = \mathbf{V}(\mathbf{S}^+)$ to be the variety generated by \mathbf{S}^+ . \mathbf{BSI} is called the variety of Boolean semilattices.¹

3. BOUNDED MORPHISMS AND INNER SUBSTRUCTURES

Let \mathbf{S} and \mathbf{T} be relational structures, and $h: S \rightarrow T$ a function. h is called a *bounded morphism* if, for each n -ary relational parameter R , the following two conditions hold:

$$\begin{aligned} & (\forall \vec{s} \in S^n) \vec{s} \in R_S \implies h\vec{s} \in R_T \\ (1) \quad & (\forall s \in S) (\forall \vec{y} \in T^{n-1}) (\vec{y}, h(s)) \in R_T \implies \\ & (\exists \vec{x} \in S^{n-1}) h\vec{x} = \vec{y} \ \& \ (\vec{x}, s) \in R_S. \end{aligned}$$

¹This is maybe not such a good name. The convention would seem to imply that a Boolean semilattice is one in which the *complex* operation \cdot is a semilattice operation.

In the special case $n = 1$, these two conditions are equivalent to

$$(\forall s \in S) s \in R_S \iff h(s) \in R_T.$$

S is called an *inner substructure* of T if the inclusion map is a bounded morphism. The conditions in (1) can be rephrased as

$$(2) \quad \begin{aligned} R_S &= R_T \cap S^n \\ (\forall s \in S)(\forall \vec{y} \in T^{n-1}) (\vec{y}, s) \in R_T &\implies \vec{y} \in S^{n-1}. \end{aligned}$$

When $n = 1$, the second condition becomes redundant.

We note for the record the following Proposition whose proof is routine.

PROPOSITION 1. *Let $h: S \rightarrow T$ be a bounded morphism, and let T' be an inner substructure of T . Then $h^{-1}(T')$ becomes an inner substructure of S when the relations are defined as in (2).*

4. A VERSION OF THE SEMILATTICE LAWS FOR TERNARY RELATIONAL STRUCTURES

Let R denote a ternary relational symbol. Consider the three laws:

- (c) $(\forall xyz) Rxyz \rightarrow Ryxz$
- (i) $(\forall x) Rxxx$
- (a) $(\forall xyzw) (\exists u) (Rxyu \ \& \ Ruzw) \leftrightarrow (\exists v) (Ryzv \ \& \ Rxvw)$

A groupoid S is commutative, idempotent or associative if and only if the associated relational structure S^\square satisfies (c), (i) and (a) respectively. Each of the three conditions is preserved by inner substructures and by bounded homomorphic images.

Semilattices

5. REPRESENTABILITY

By a *Boolean groupoid* we mean a Boolean algebra with a single binary operator that is normal and additive. We say that a Boolean groupoid B is *semilattice-representable* if $B \in \mathbf{ISP}(S^+)$.

6. DOWNSETS

Let B be a Boolean groupoid and $x \in B$. We define $\downarrow x$ to be $x \cdot 1$. If S is a semilattice and $X \in S^+$ then it follows that $\downarrow X = \{s \in S : \exists x \in X s \preceq x\}$, in other words, the downset of S generated by X .

7. BOUNDED HOMOMORPHISMS OF A SEMILATTICE

Let $\mathbf{S} \in \mathbf{S}$ and $h: \mathbf{S}^\square \rightarrow \langle U, R_U \rangle$ a surjective, bounded homomorphism. Let $\theta = \ker h$. Then

- (1) $R_U = \{ (hx, hy, hz) : (x, y, z) \in R_S \}$, and
- (2) For all $a, b \in S$, the image of the map $a/\theta \times b/\theta \rightarrow S$ given by $(x, y) \mapsto x \cdot y$ is a union of θ -classes.

Conversely, if θ is an equivalence relation on S satisfying (2), then by setting $U = S/\theta$ and defining R_U as in (1), the natural map $S \rightarrow U$ is a bounded homomorphism.

The proof is a straightforward verification. More generally, if \mathbf{S} is only an inner substructure of a semilattice, the above characterization still holds if we replace condition (2) with

- (2') For all $(a, b, c) \in R_S$, the image of the partial map $a/\theta \times b/\theta \rightarrow S$ given by $(x, y) \mapsto x \cdot y$ is a union of θ -classes.

We shall call an equivalence relation *bounded* if it satisfies the condition in (2').

We will also be interested in semilattices with an additional constant symbol c . According to ¶3, h must also satisfy, for each $x \in S$, $x = c \iff h(x) \in c_U$. (Here, c_U denotes the unary relation on U corresponding to the additional constant.) In terms of θ we have

$$c/\theta = \{c\}.$$

8. CORE EXAMPLE OF A BOUNDED HOMOMORPHIC IMAGE OF A SEMILATTICE

Let \mathbf{S} be the semilattice with universe $\{a, b_1, b_2\}$ and $a = b_1 \cdot b_2$. (The free semilattice on two generators, as it happens.) Let θ be the equivalence relation that identifies b_1 and b_2 . Then θ is a bounded equivalence on \mathbf{S} (see ¶7). Thus the mapping $\mathbf{S} \rightarrow \mathbf{S}/\theta$ is a bounded homomorphism. But the object \mathbf{S}/θ is not a semilattice. (Nor is it a disjoint union, nor an inner substructure of a semilattice.)

9. BOUNDED HOMOMORPHISMS BETWEEN SEMILATTICES

A bounded homomorphism from one semilattice onto another is a semilattice homomorphism.

10. THE INNER SUBSTRUCTURES OF A SEMILATTICE

Let $\mathbf{S} \in \mathbf{S}$. The inner substructures of \mathbf{S}^\square are precisely the upsets of \mathbf{S} . That is: $\langle U, R|_U \rangle$ is an inner substructure of $\langle S, R \rangle$ if and only if

$$U = \uparrow U = \{ s \in S : \exists u \in U \ s \succeq u \}.$$

PROOF: Let U be an inner substructure of \mathbf{S} . We wish to show U is an upset. So let $u \in U$ and $u \preceq s$. Then $u \cdot s = u$, in other words, $(u, s, u) \in R$. By [Goldblatt, 3.2.2], this implies that $s \in U$.

Conversely, suppose U is an upset. If $(x, y, z) \in R$ and $z \in U$, then $z = x \cdot y \preceq x, y$ implies that $x, y \in U$ since U is an upset. Thus U is an inner substructure.

11. AN AXIOMATIZATION OF UPSETS

Let \cdot be a partial binary operation on a set T satisfying:

$$x \cdot x = x, \quad x \cdot y = y \cdot x, \quad x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

where, for any $x, y, z \in T$ one side is defined if and only if the other side is defined. Then $\langle T, \cdot \rangle$ is an upset of a meet semilattice.

PROOF: It is easy to check that the relation $x \preceq y \iff x \cdot y = x$ is a partial ordering of T . Let $S = T \cup \{\perp\}$, and extend the ordering so that $\perp \preceq x$ for all $x \in T$. Then $\langle S, \preceq \rangle$ becomes a semilattice, and T is an upset.

12. DISJOINT UNIONS OF SEMILATTICES

Let $S_i \in \mathbf{S}$ for $i \in I$. Then the relational structure $\bigcup_{i \in I} S_i^\square$ is an inner substructure of a member of \mathbf{S} .

PROOF: This is a special case of the construction in ¶11. Just adjoin a new least element to the disjoint union.

13. THE CANONICAL EXTENSION OF A BOOLEAN GROUPOID

Let \mathbf{B} be a Boolean groupoid. The *canonical extension* of \mathbf{B} is the Boolean algebra \mathbf{B}^σ as defined by Jónsson and Tarski. \mathbf{B}^σ is a complete and atomic Boolean algebra with atom structure \mathbf{B}_+^σ . B_+^σ can be thought of as the set of ultrafilters of \mathbf{B} . The extension is made into a Boolean groupoid by defining, for $a, b \in B_+^\sigma$

$$a \cdot b = \bigwedge \{x \cdot y : a \leq x, b \leq y, x, y \in B\}.$$

14. THE CANONICAL EXTENSION OF A BSL

Let \mathbf{S} be a semilattice. Then the canonical extension of \mathbf{S}^+ is an algebra \mathbf{T}^+ . What is the structure $\mathbf{T} = \langle T, R_T \rangle$? (In Goldblatt's terminology, \mathbf{T} is called the canonical extension, and \mathbf{T}^+ is the canonical embedding algebra.)

The set T is the set of all ultrafilters over the set S . Now in the embedding $\varphi: \mathbf{S}^+ \hookrightarrow \mathbf{T}^+$, the image of a complex A (of \mathbf{S}) is the join of all of the atoms of \mathbf{T}^+ that lie below it. But if U is an ultrafilter over S , then $\{U\} \leq \varphi(A)$ if and only if $A \in U$. In other words, $\varphi(A)$ can be identified with the set of ultrafilters that contain A .

Now let U and V be ultrafilters over S (that is to say, atoms of \mathbf{T}^+). The definition of the groupoid operation on \mathbf{T}^+ and the above considerations imply that

$$\{U\} \cdot \{V\} = \bigcap \{ \varphi(A \cdot B) : A \in U \text{ \& } B \in V \}.$$

Finally, we can describe the ternary relation R_T as follows. For any three ultrafilters U, V, W over S ,

$$(U, V, W) \in R_T \iff \{W\} \leq \{U\} \cdot \{V\} \iff W \supseteq \{ (A \cdot B) : A \in U \text{ \& } B \in V \}.$$

Note that the set $\{(A \cdot B) : A \in U \text{ \& } B \in V\}$ always has the finite intersection property. For consider any intersection $X = (A_1 \cdot B_1) \cap \cdots \cap (A_n \cdot B_n)$. Let $A = A_1 \cap \cdots \cap A_n \in U$, and $B = B_1 \cap \cdots \cap B_n \in V$. Then, A and B are nonempty, and, since they are complexes of a semilattice, $A \cdot B$ is nonempty as well. But by monotonicity, $A \cdot B \subseteq X$, so $X \neq \emptyset$.

[Here is another way to look at the above paragraph. S^+ is integral, and the sentence $x \neq 0 \text{ \& } y \neq 0 \rightarrow x \cdot y \neq 0$ is preserved by canonical extensions.]

15. A QUICK REVIEW OF THE DUALITY BETWEEN FRAMES AND BAOs

Let $\langle S, R_S \rangle$ and $\langle T, R_T \rangle$ be frames (i.e., sets with a ternary relation), and $h: S \rightarrow T$ a bounded morphism. Then there is an induced bao-homomorphism $h^+: T^+ \rightarrow S^+$ given by $h^+(X) = h^{-1}(X)$. If h is injective, then h^+ is surjective, and if h is surjective, then h^+ is injective.

Now let \mathbf{B} and \mathbf{C} be Boolean groupoids. We define \mathbf{B}_δ to be the frame on \mathbf{B}_+^σ . Let $k: \mathbf{B} \rightarrow \mathbf{C}$ be a homomorphism. Then there is a bounded morphism $k_\delta: \mathbf{C}_\delta \rightarrow \mathbf{B}_\delta$ given by $k_\delta(c) = \bigwedge \{b \in B^\sigma : c \leq k^\sigma(b)\}$. Once again, we have that if k is injective (surjective), then k_δ is surjective (injective).

16. FUNCTIONALITY IN RELATIONAL STRUCTURES

Call a $n+1$ -ary relation *functional* if it is the graph of an n -ary function. Functionality is not preserved by bounded homomorphic images. However injectiveness is preserved. That is, each of these properties is preserved by bounded homomorphic images:

- Inj1. $Rxy \text{ \& } Rx'y \rightarrow x = x'$
 Inj2. $Rxyz \text{ \& } Rx'yz \rightarrow x = x'$.

Several properties, in conjunction with functionality, imply injectivity. For example

$$\begin{aligned} (\forall x) Rxx \text{ \& } R \text{ functional} &\implies \text{Inj1} \\ (\forall x, y) (Rxy \rightarrow Ryx) \text{ \& } R \text{ functional} &\implies \text{Inj1} \\ (\forall x, y, z) (Rxyz \rightarrow Rzyx) \text{ \& } R \text{ functional} &\implies \text{Inj2} \end{aligned}$$

The first two of these imply that the class of all reflexive, and of all symmetric binary relational structures can not be obtained as the bounded homomorphic images of a class of mono-unary algebras. (Which is pretty easy to see directly.) From the third implication we conclude that there are ternary relational structures satisfying $Rxyz \rightarrow Rzyx$ that are can not be represented as the bounded homomorphic image of a groupoid.

For some reason that I can no longer remember, I thought that complex algebras of squags might be relevant here.

17. SOME GENERAL REMARKS ABOUT FRAMES BUILT FROM ALGEBRAS

Suppose \mathbf{A} is a partial algebra, say a partial groupoid. Create the groupoid $\bar{\mathbf{A}}$ with $\bar{A} = A \cup \{\infty\}$ and define $x \cdot y = \infty$ in $\bar{\mathbf{A}}$ if it is undefined in \mathbf{A} . Then it is easy to check that \mathbf{A} is an inner substructure of $\bar{\mathbf{A}}$. (The construction in ¶11 is a special case of this.) What properties of \mathbf{A} are preserved when passing to $\bar{\mathbf{A}}$?

Let \mathbf{M} be the class of mono-unary algebras, and \mathbf{B} the class of structures with one binary relation. Then it is not hard to show that every member of \mathbf{B} is a bounded homomorphic image of an inner substructure of a member of \mathbf{M} . Therefore $\mathbf{V}(\mathbf{B}^+) = \mathbf{V}(\mathbf{M}^+)$.

There is an analogous argument for the classes of all groupoids and of all ternary relational structures.

2. THE ALGEBRAIC THEORY OF BOOLEAN SEMILATTICES

18. AXIOMS

One of our objectives is to axiomatize BSl. Let Σ denote the following set of axioms in the language of Boolean groupoids $\langle B, \vee, \wedge, ', 0, 1, \cdot \rangle$.

1. Axioms for a Boolean algebra
2. $x \cdot 0 = 0$
3. $x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z)$
4. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
5. $x \cdot y = y \cdot x$ (Σ)
6. $x \leq x \cdot x$
7. $x \cdot y \cdot 1 = (x \cdot 1) \wedge (y \cdot 1)$
8. $x \wedge (y \cdot 1) \leq x \cdot y$
9. $x \cdot y = x \vee z \rightarrow x \cdot z \leq x^2 \vee z^2$.

x^2 is shorthand for $x \cdot x$. Observe that axioms 1–8 are identities, while 9 is only universal Horn. However, it is not hard to show that 9 can be replaced by an identity (see ¶19).

THEOREM 2. $S^+ \models \Sigma$.

The proof is a straightforward verification.

19. EQUIVALENT FORMS OF Σ

Σ_8 (that is, axiom 8 of Σ , see ¶18) can be replaced by

$$8'. \quad x \leq y \cdot 1 \rightarrow x \leq y \cdot x$$

which is useful in practice. There is also a symmetric version of Σ_8 :

$$8''. \quad (x \cdot 1) \wedge (y \cdot 1) \wedge (x \vee y) \leq x \cdot y.$$

Σ_6 can be replaced by

$$6'. \quad x \wedge y \leq x \cdot y.$$

Axiom 9 can be replaced by

$$9'. \quad x \cdot ((x \cdot 1) - x) \leq x^2 \vee ((x \cdot 1) - x)^2$$

where $x - y$ is shorthand for $x \wedge y'$.

PROOF: Let \mathbf{B} be a Boolean groupoid and suppose that $\mathbf{B} \models \Sigma$. To verify that $\Sigma_{8'}$ holds, suppose that $x, y \in B$ and $x \leq y \cdot 1$. Then $x = x \wedge (y \cdot 1)$, so by Σ_8 , $x \leq x \cdot y$. To check $\Sigma_{6'}$, observe that by axioms 1, 3, 6, $y \leq y \cdot y \leq y \cdot 1$. Therefore $x \wedge y \leq x \wedge (y \cdot 1) \leq x \cdot y$ by 8.

Now suppose that we replace 8 with $8'$ in Σ . To derive Σ_8 , let $z = x \wedge y \cdot 1$. Then $z \leq y \cdot 1$, so by $\Sigma_{8'}$, $z \leq y \cdot z$. But $z \leq x$, so $z \leq y \cdot z \leq y \cdot x$ (by additivity), thus Σ_8 holds. Σ_6 follows easily from $\Sigma_{6'}$ by taking $x = y$. Finally, Σ_8 and $\Sigma_{8''}$ are easily seen to be equivalent by applying distributivity and the fact that $x \leq x \cdot x \leq x \cdot 1$.

We see that $\Sigma \vdash \Sigma_{9'}$ by taking $y = 1$ and $z = (x \cdot 1) - x$ in Σ_9 . For the converse, suppose that $x \cdot y = z \vee z$. Let

$$w = (x \cdot y) - x. \text{ Then } w \leq (x \cdot 1) - x \text{ and}$$

$$x \cdot z = x \cdot (w \vee (x \wedge z)) = x \cdot w \vee x \cdot (x \wedge z) \leq x \cdot ((x \cdot 1) - x) \vee x \cdot x \leq x^2 \vee (x \cdot 1 - x)^2$$

the last inequality coming from $\Sigma_{9'}$.

COROLLARY 3. $\mathbf{BSI} \models \Sigma$.

The closure operation

20. A CLOSURE OPERATION

THEOREM 4. If $\mathbf{B} \models \Sigma$, then \downarrow is a closure operation on the Boolean algebra \mathbf{B} .

PROOF: Follows from axioms 3, 4 and 6 of Σ .

21. A FUNCTOR FROM \mathbf{BSI} TO CLOSURE ALGEBRAS

Let $\mathbf{B} \models \Sigma$. Define $\mathbf{B}^\downarrow = \langle B, \wedge, \vee, ', 0, 1, \downarrow \rangle$ to be the reduct of \mathbf{B} obtained by replacing complex product with \downarrow . Then \mathbf{B}^\downarrow is a closure algebra. The assignment $\mathbf{B} \mapsto \mathbf{B}^\downarrow$ is a covariant functor from the category \mathbf{BSI} to the category of closure algebras.

22. GENERATING THE VARIETY OF ALL CLOSURE ALGEBRAS

THEOREM 5. $\mathbf{V}\{(S^+)^{\downarrow} : S \in \mathcal{S}\}$ is the variety of all closure algebras.

PROOF: Let S be a complete, rooted, countably infinite binary tree. The closure algebra $S^{+\downarrow}$ is identical to the complex algebra of the poset S . It is a theorem of Blok's, see [??]² that this closure algebra generates the whole variety.

23. AN ATTEMPT TO RETREIVE THE SEMILATTICE ORDER FROM A COMPLEX ALGEBRA

Let $\mathbf{B} \models \Sigma$. For $x, y \in B$ define

$$x \trianglelefteq y \iff x \cdot 1 \leq y \cdot 1.$$

Then ' \trianglelefteq ' is clearly a quasiorder on B . Let us define the induced equivalence relation:

$$x \sim y \iff x \cdot 1 = y \cdot 1.$$

Notice that if $\mathbf{B} = S^+$ for a semilattice S , then the *atoms* of \mathbf{B} correspond to the points of S , and for any $x, y \in S$, we have $x \leq y$ iff $\{x\} \trianglelefteq \{y\}$. Thus it would be useful to know if the poset induced on B/\sim or on B_+/\sim has any nice properties for an arbitrary Boolean semilattice.

We do have the following observation. Let $\mathbf{B} \models \Sigma$. Then

$$x_1 \sim x_2 \ \& \ y_1 \sim y_2 \implies x_1 \cdot y_1 \sim x_2 \cdot y_2 \ \& \ x_1 \vee y_1 \sim x_2 \vee y_2.$$

These follow easily from axioms Σ_7 , Σ_4 and Σ_5 . Therefore ' \sim ' is a congruence relation on the $\{\vee, \cdot, 0, 1\}$ -reduct of \mathbf{B} .

In fact, let $\text{Idl}(\mathbf{B})$ denote the lattice of congruence ideals of \mathbf{B} (which is, of course, isomorphic to the lattice of congruence relations on \mathbf{B}). Define a function from B to $\text{Idl}(\mathbf{B})$ by $x \mapsto (\downarrow x]$. Obviously, the kernel of this map is the equivalence relation ' \sim ', and the induced map

$$\langle B/\sim, \vee, \cdot, 0, 1 \rangle \longrightarrow \text{Idl}(\mathbf{B})$$

is an injective homomorphism. Thus the structure on the left is isomorphic to the distributive lattice of principal congruence ideals of \mathbf{B} . In particular, \mathbf{B} is subdirectly irreducible if and only if \mathbf{B}/\sim has least non-zero element under \trianglelefteq .

²How about a reference, Wim?

24. HOW MUCH RICHER THAN CLOSURE ALGEBRAS ARE BOOLEAN SEMILATTICES?

We know that the variety generated by the closure-reducts of all Boolean semilattices is the variety of all closure algebras. (Theorem 5.) Is it true that every closure algebra is the reduct of a BSL? There are also examples of non-isomorphic Boolean semilattices with the same closure algebra reduct—the algebras \mathbf{A}_1 , \mathbf{A}_3 and \mathbf{A}_4 of article 43 have isomorphic closure structure. Nevertheless, it seems to me that the general complex product does not contain too much more information than does the closure operation. Let $\langle P, \preceq \rangle$ be a poset (or even a qoset). Define a ternary relation

$$R' = \{ (x, y, z) \in P^3 : z \preceq x, y \}.$$

Let us consider the Boolean groupoid $\mathbf{B} = \langle P, R' \rangle^+$. One easily checks that $\mathbf{B} \models \Sigma$. Furthermore, \mathbf{B} satisfies $x \cdot y = x \cdot y \cdot 1 = \downarrow x \wedge \downarrow y$. I'm not sure what I am getting at here. One could look at the subvariety defined by this latter identity.

25. A STRONG FORM OF CEP

Let $\mathbf{B} \in \text{BSI}$, and let \mathbf{C} be a subalgebra of \mathbf{B}^\downarrow (that is, \mathbf{C} is a closure algebra, and a subalgebra of the \downarrow -reduct of \mathbf{B}). Let $\theta \in \text{Con } \mathbf{C}$. Verify that there is $\bar{\theta} \in \text{Con } \mathbf{B}$ such that the diagram commutes:

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{\text{bsl}} & \mathbf{B}/\bar{\theta} \\ \uparrow & & \uparrow \\ \mathbf{C} & \longrightarrow & \mathbf{C}/\theta \end{array}$$

where the top map is a Boolean semilattice homomorphism, the other three are closure algebra maps.

From this it follows that if \mathbf{V} is a variety of Boolean semilattices, then $\mathbf{V}^\downarrow = \{ \mathbf{B}^\downarrow : \mathbf{B} \in \mathbf{V} \}$ is a variety of closure algebras.

Problem 1. If $\mathbf{K} \subseteq \mathbf{S}$, does it follow that $\mathbf{V}(\mathbf{K}^+)^{\downarrow} = \mathbf{V}(\mathbf{K}^{\downarrow})^+$?

Let \mathbf{S} be the three element nonlinear semilattice. Can ' \cdot ' be retrieved from ' \downarrow ' in $\mathbf{V}(\mathbf{S}^+)$?

26. ' \uparrow ' IS NOT A TERM OF BSL

Let $S = \{a, b, c, d\}$ ordered by: $d \preceq a, b \preceq c$, and let $X = \{a\}$. Then the subalgebra of \mathbf{S}^+ generated by X is the eight-element algebra with atoms: $\{a\}, \{d\}, \{b, c\}$. Since $\uparrow X = \{a, c\}$ is not a member of that subalgebra, ' \uparrow ' is not a term operation, even on the class $\mathbf{S}(\mathbf{S})$.

We define the interior operation $x^\circ := (x' \cdot 1)'$. For a complex X of a semilattice \mathbf{S} , X° will be the largest upset of \mathbf{S} contained in X .

27. AN OBVIOUS FACT ABOUT THE INTERIOR OPERATION

$\Sigma \vdash (x \wedge y)^\circ = x^\circ \wedge y^\circ$. (See article 26.)

PROOF: $(x \wedge y)^\circ = (\downarrow((x \wedge y)'))' = (\downarrow(x' \vee y'))' = (\downarrow x' \vee \downarrow y')' = (\downarrow x')' \wedge (\downarrow y')' = x^\circ \wedge y^\circ$.

Subdirectly irreducible algebras

28. CONGRUENCE IDEALS

Let \mathbf{B} be any bao and θ a congruence of \mathbf{B} . The congruence class $0/\theta$ is, of course, a (Boolean) ideal of \mathbf{B} . However, it will generally have other special properties. As is customary, we make the following definition.

DEFINITION 6. An ideal I of a bao \mathbf{B} is called a *congruence ideal* if there is $\theta \in \text{Con}(\mathbf{B})$ such that $I = 0/\theta$.

As is well-known, there is a 1-1 correspondence between congruences and congruence ideals in any bao.

29. A CHARACTERIZATION OF CONGRUENCE IDEALS

THEOREM 7. (Jipsen, [??, Lemma 2.1]) *Let \mathbf{B} be a Boolean groupoid and I an ideal of \mathbf{B} . Then I is a congruence ideal if and only if $x \in I \implies x \cdot 1, 1 \cdot x \in I$.*

COROLLARY 8. *Let \mathbf{B} be a Boolean groupoid and $\mathbf{B} \models \Sigma$.*

- (1) *If I is an ideal of \mathbf{B} then I is a congruence ideal if and only if $x \in I \implies \downarrow x \in I$.*
- (2) *If $a \in B$, then the smallest congruence ideal containing a is*

$$(\downarrow a] = \{x \in B : x \leq \downarrow a\}.$$

An element a such that $(a]$ is a congruence ideal is usually called a *congruence element*. It follows from the above Corollary, that the congruence elements are precisely the closed (under \downarrow) elements of B . If \mathbf{S} is a semilattice, then the congruence elements of \mathbf{S}^+ are precisely the downsets of \mathbf{S} .

30. SUBDIRECT IRREDUCIBILITY AND CANONICAL EXTENSIONS

THEOREM 9. *If \mathbf{A} is a subdirectly irreducible model of Σ , then \mathbf{A}^σ is subdirectly irreducible.*

PROOF: \mathbf{A} has a smallest nontrivial ideal, which in turn must be generated by a single closed element a . More precisely, $a = a \cdot 1$ and, for every $x > 0$, $x \cdot 1 \geq a$. Now \mathbf{A} is a subalgebra of \mathbf{A}^σ so $a \cdot 1 = a$ continues to hold in the larger algebra.

Let y be an atom of \mathbf{A}^σ . By the definition of the canonical extension,

$$y \cdot 1 = \bigwedge \{x \cdot 1 : y \leq x \in A\} \geq a.$$

Therefore, a generates the monolith of \mathbf{A}^σ .

31. THE PRODUCT OF CONGRUENCE IDEALS

Let \mathbf{B} be a Boolean groupoid, $\mathbf{B} \models \Sigma$. If I and J are congruence ideals of \mathbf{B} , then $I \cap J = (I \cdot J]$. (Here, $I \cdot J = \{x \cdot y : x \in I, y \in J\}$, which is a bit of an abuse of notation.)

PROOF: If $x \in I \cap J$, then $x \leq x \cdot x \in I \cdot J$, so $x \in (I \cdot J]$. Conversely, if $x \in I$ and $y \in J$, then $x \cdot y \leq x \cdot 1 \in I$ (since I is a congruence ideal). Similarly, $x \cdot y \in J$. Therefore $I \cdot J \subseteq I \cap J$. Since $I \cap J$ is an ideal, we conclude $(I \cdot J] \subseteq I \cap J$.

32. A CHARACTERIZATION OF SIMPLICITY AND SUDIRECT IRREDUCIBILITY

Let \mathbf{B} be a Boolean groupoid. \mathbf{B} is *integral* if $x, y \neq 0 \implies x \cdot y \neq 0$.

THEOREM 10. Let $\mathbf{B} \models \Sigma$. Then \mathbf{B} is simple iff $x \neq 0 \implies \downarrow x = 1$. \mathbf{B} is finitely subdirectly irreducible if and only if it is integral.

PROOF: The characterization of simplicity follows from Corollary 8, while that of finite subdirect irreducibility follows from article 31.

COROLLARY 11. Let \mathbf{S} be a semilattice. Then \mathbf{S}^+ is simple iff $|S| = 1$, and \mathbf{S}^+ is subdirectly irreducible iff \mathbf{S} has a least element.

PROOF: If $|S| = 1$, then $|S^+| = 2$, so \mathbf{S}^+ is obviously simple. Conversely, if a and b are distinct elements of S , with, say, $a \not\leq b$, then $a \notin \downarrow b$, so that, by Theorem 10, \mathbf{S}^+ is not simple.

If \mathbf{S} has a least element \perp , then $\downarrow\{\perp\} = \{\perp\}$ in \mathbf{S}^+ , so $(\{\perp\})$ is the least non-trivial congruence ideal of \mathbf{S}^+ . Conversely, suppose \mathbf{S}^+ has such a least nontrivial congruence ideal, M . Then there is a complex (of S) $X \in M$, $X \neq \emptyset$. Let $a \in X$. If a is not the least element of \mathbf{S} , then there is $b \leq a$ (since \mathbf{S} is a meet semilattice). Therefore, $(\downarrow b)$ is a congruence ideal properly contained in M , which is a contradiction.

Notice that any algebra of the form \mathbf{S}^+ (for $\mathbf{S} \in \mathbf{S}$) is integral. Therefore, for any $\mathbf{S} \in \mathbf{S}$, \mathbf{S}^+ is finitely subdirectly irreducible.

33. A REMARK ABOUT SIMPLE ALGEBRAS

Every simple algebra satisfies

$$x > 0 \ \& \ y > 0 \rightarrow x \cdot y \geq x \vee y.$$

This follows immediately from Σ_8 and Theorem 10.

34. A NECESSARY CONDITION FOR FINITE REPRESENTABILITY

THEOREM 12. Let $\mathbf{B} \models \Sigma$, $r \in B$. Suppose that $r \cdot 1 = 1$. Then for any homomorphism $\alpha: \mathbf{B} \rightarrow \mathbf{S}^+$, for a semilattice \mathbf{S} , the complex $\alpha(r)$ must contain all maximal elements of \mathbf{S} .

PROOF: Let $R = \alpha(r) \subseteq S$. Then $r \cdot 1 = 1$ implies that $R \cdot S = S$ in S^+ . Let u be a maximal element of S . Then $u \in R \cdot S$ implies that $u = x \cdot y$ for some $x \in R$, $y \in S$. Consequently, $u \preceq x$. By maximality, $u = x \in R$.

COROLLARY 13. *Let $\mathbf{B} \models \Sigma$. Suppose there is an element $r \in B$ such that $r \cdot 1 = r' \cdot 1 = 1$. Then there is no homomorphism from \mathbf{B} to S^+ for any semilattice S with a maximal element. In particular, no semilattice representation of \mathbf{B} involves a finite semilattice.*

PROOF: Let $\alpha: \mathbf{B} \rightarrow S^+$ be a homomorphism. By the Theorem, both $\alpha(r)$ and $\alpha(r')$ must contain every maximal element of S . Since these two sets are disjoint, S has no maximal elements.

COROLLARY 14. *No simple member of $\text{Mod}(\Sigma)$ is finitely semilattice-representable.*

PROOF: Follows from Theorem 10 and the previous Corollary.

Let us remark that Theorem 12 also holds when S is an upset of a semilattice. By ¶10, these are the inner substructures. Thus the statements in this article apply more generally to homomorphic images of members of S^+ .

35. A CONGRUENCE ASSOCIATED WITH AN IDENTITY ELEMENT

Let $\mathbf{A} \models \Sigma$ and assume that \top is an identity on \mathbf{A} . Then $\downarrow \top = \top \cdot 1 = 1$. But more interesting: $\downarrow(\top') = \top'$, in other words, it is a congruence element.

The proof is easy. By Σ_8 , $\top \wedge (\top' \cdot 1) \leq \top' \cdot \top = \top'$. On the other hand, $\top \wedge (\top' \cdot 1) \leq \top$. It follows that $\top' \cdot 1 \leq \top'$, and therefore, \top' is closed.

3. SUBVARIETIES OF BOOLEAN SEMIALTTICES

36. EQUATIONALLY DEFINABLE PRINCIPAL CONGRUENCES

THEOREM 15. *The variety $\text{Mod}(\Sigma)$ has equationally definable principal congruences.*

PROOF: From standard facts about Boolean algebras, $c \equiv d \pmod{\theta(a, b)}$ if and only if $c \oplus d \equiv 0 \pmod{\theta(a \oplus b, 0)}$. From Corollary 8, this latter condition is equivalent to $c \oplus d \leq \downarrow(a \oplus b)$.

37. COMPACT CONGRUENCES ON COMPLEX ALGEBRAS

In EDPC I, Blok and Pigozzi prove that, under EDPC:

$$\mathbf{V}(\mathbf{K})_{\text{fsi}} \subseteq \mathbf{SH}_{\omega}(\mathbf{K})$$

for a class \mathbf{K} . This suggests that we should look at compact congruences on complex algebras.

A congruence θ on a Boolean algebra \mathbf{B} is compact if and only if $0/\theta = (a]$ for some element $a \in B$. Therefore, $\mathbf{B}/\theta \cong (a']$ when the latter is viewed as a relative Boolean algebra. If \mathbf{B} is complete and atomic, so is $(a']$. It follows that if \mathbf{B} is a BAO, then $\mathbf{H}_\omega(\mathbf{B})$ consists of complex algebras.

In particular, suppose \mathbf{S} is a semilattice, and θ a compact congruence on \mathbf{S}^+ . Then $0/\theta = (D]$ for some downset D of \mathbf{S} , and $\mathbf{S}^+/\theta \cong (D'] \cong D'^+$, i.e., the complex algebra of the upset (inner substructure) D' .

38. IMPROVED DISCRIPTION OF THE VARIETY BSI

From ¶12 and duality, we obtain: $\mathbf{P}(\mathbf{S}^+) \subseteq \mathbf{H}(\mathbf{S}^+)$. By item 36, the variety BSI has EDPC. Consequently, it has the congruence extension property. Combining this with the above inclusion we have:

$$\mathbf{BSI} = \mathbf{HSP}(\mathbf{S}^+) = \mathbf{HS}(\mathbf{S}^+) = \mathbf{SH}(\mathbf{S}^+).$$

39. THE QUASIVARIETY GENERATED BY \mathbf{S}

Problem 2. Is $\mathbf{BSI} = \mathbf{ISPP}_u(\mathbf{S}^+)$?

40. THE FINITE BASIS QUESTION

Problem 3. Is the variety BSI finitely based?

Related Varieties

41. ADJOINING A LARGEST ELEMENT TO \mathbf{S}

Let \mathbf{S}_\top denote the variety of semilattices with a largest element, \top . The language of the complex algebras of \mathbf{S}_\top contains a new nullary operation symbol \top whose interpretation in a complex algebra is, of course, $\{\top\}$.

THEOREM 16. *If Γ is an equational base for BSI then $\Gamma_\top = \Gamma \cup \{x \cdot \top = x\}$ is an equational base for $\mathbf{V}(\mathbf{S}_\top^+)$.³*

PROOF: Certainly $\mathbf{V}(\mathbf{S}_\top^+) \models \Gamma_\top$. Thus it suffices to show that if \mathbf{A}_\top is a subdirectly irreducible Boolean groupoid with an additional constant operation \top , and $\mathbf{A}_\top \models \Gamma_\top$, then $\mathbf{A}_\top \in \mathbf{V}(\mathbf{S}_\top^+)$. We let \mathbf{A} denote the reduct of the algebra \mathbf{A}_\top obtained by omitting the name for the constant \top .

Since $\mathbf{A}_\top \models \Gamma_\top$, we certainly have $\mathbf{A} \models \Gamma$ and $\mathbf{A} \models (\exists y) x \cdot y = x$. Therefore, $\mathbf{A} \in \mathbf{BSI}$ by assumption. Since \mathbf{S} is an elementary class, it follows from Golblatt's results [??, 3.6.3] that BSI is a canonical variety. In other words, $\mathbf{A} \in \mathbf{BSI}$ implies $\mathbf{A}^\sigma \in \mathbf{BSI}$.

Now, by ¶30, \mathbf{A}^σ is subdirectly irreducible. We have already observed that $\mathbf{A} \models (\exists y) x = x \cdot y$. Since both sides of the equation are positive terms, it is preserved under canonical extensions. Therefore, \mathbf{A}^σ satisfies the same sentence, in other words, \mathbf{A}^σ has an identity element.

³Converse?

Claim: *The identity element e , of \mathbf{A}^σ is an atom.*

PROOF: Suppose that $0 < a < e$ on \mathbf{A}^σ . Let $b = e \wedge a'$. Then $0 < b < e$. We have $a = a \cdot e = a \cdot (a \vee b) = a^2 \vee (a \cdot b)$ which implies that $a \geq a \cdot b$. Similarly, $b \geq a \cdot b$, and therefore, $0 = a \wedge b \geq a \cdot b$. But \mathbf{A}^σ is subdirectly irreducible, hence integral. This contradiction establishes the claim.

By article 38, $\text{BSI} = \mathbf{HS}(\mathbf{S}^+)$. Thus, there is a semilattice $\mathbf{S} \in \mathbf{S}$, a subalgebra \mathbf{B} of \mathbf{S}^+ and a surjective (BAO) homomorphism $k: \mathbf{B} \rightarrow \mathbf{A}$. Applying the duality outlined in article 15, we obtain bounded morphisms in the opposite directions:

$$\begin{array}{ccc} \mathbf{S}^+ & \longleftarrow & \mathbf{B} \\ & & \downarrow k \\ & & \mathbf{A} \end{array} \qquad \begin{array}{ccccc} \mathbf{T} & \longrightarrow & \mathbf{S}_\delta^+ & \longrightarrow & \mathbf{B}_\delta \\ & & & & \uparrow k_\delta \\ & & & & \mathbf{A}_\delta \end{array}$$

Furthermore, by Goldblatt's theorem [??, 3.6.1], there is an ultrapower \mathbf{T} of \mathbf{S} and a surjective bounded morphism from \mathbf{T} to \mathbf{S}_δ^+ . (In Goldblatt's terminology, \mathbf{S}_δ^+ is the *canonical extension* of \mathbf{S} .) Since \mathbf{S} is an elementary class, \mathbf{T} is a semilattice.

Let h be the surjective, bounded morphism from \mathbf{T} to \mathbf{B}_δ obtained above, and let $\theta = \ker h$. Then θ is a bounded equivalence relation (see article 7). Let $C = h^{-1}(k_\delta^{-1}(A_\delta))$. By choosing the ternary relation on C to be the restriction of that on \mathbf{T} , we can easily check that \mathbf{C} becomes an inner substructure of \mathbf{T} . By article 10, this means that C is an upset of the semilattice \mathbf{T} .

Since e is an atom of \mathbf{A}^σ , we have $e \in A_\delta$. Let $E = h^{-1}(e)$. Certainly $E \subseteq C$. Notice also that E is a single θ -class.

Claim: *E is an upset of \mathbf{T} .*

PROOF: Let $x \in E$ and $x \preceq y \in T$. Since C is an upset, $y \in C$. Now $x \cdot y = x$ in \mathbf{T} , and h is bounded, so $(hx, hy, hx) \in R_{A_\delta}$. But $hx = e$, so in \mathbf{A}^σ this yields, $e \cdot hy \geq e$. But e is an identity of \mathbf{A}^σ and hy is an atom, so $hy = e$, in other words, $y \in E$.

Let \top denote a point not in T , and define $\hat{T} = T - E \cup \{\top\}$. We order \hat{T} by setting $\top > x$ for all $x \in T - E$. Since E was an upset it follows that \hat{T} is a semilattice with largest element. By expanding the type to include a name for the new constant, we make \hat{T} into a member of \mathbf{S}_\top .

Finally, define $\hat{\theta} = (\theta \cup \{(\top, \top)\}) \cap \hat{T}$. $\hat{\theta}$ is an equivalence relation on \hat{T} . The equivalence classes of $\hat{\theta}$ are the same as those of θ except that the class E of θ has been replaced by the singleton $\{\top\}$ in $\hat{\theta}$.

Claim: *The natural map from \hat{T} to $\hat{T}/\hat{\theta}$ is a bounded morphism of type $\langle 3, 1 \rangle$.*

PROOF: We need to prove that $\hat{\theta}$ is a "bounded equivalence relation" of type $\langle 3, 1 \rangle$ (see ¶7.) For the unary relation, the requirement is that $\top/\hat{\theta} = \{\top\}$ which we

observed above. Regarding the ternary relation, we need to show that for any $x, y \in \widehat{T}$, $x/\hat{\theta} \cdot y/\hat{\theta}$ is a union of $\hat{\theta}$ -classes.

First, suppose that neither x nor y is equal to \top . Then

$$x/\hat{\theta} \cdot y/\hat{\theta} = x/\theta \cdot y/\theta = \bigcup_{z \in Z} z/\theta$$

for some set Z , since θ is bounded. Now $z/\theta = z/\hat{\theta}$ as long as $z \notin E$. But $z = x_1 \cdot y_1$ for some $x_1 \theta x$ and $y_1 \theta y$, so, $z \preceq y_1$. If $z \in E$, then, since E is both an upset and a θ -class, $y_1 \in E$, so $y \in E$ which is false.

On the other hand, if say, $x = \top$, then

$$x/\hat{\theta} \cdot y/\hat{\theta} = \{\top\} \cdot y/\hat{\theta} = y/\hat{\theta},$$

trivially a union of $\hat{\theta}$ -classes.

Now $\widehat{\mathbf{T}}/\hat{\theta} \cong \mathbf{T}/\theta$ since we haven't created any new equivalence classes. It follows that \mathbf{A}_δ is a bounded morphic image of an inner substructure of a member of \mathbf{S}_\top . Applying duality, $\mathbf{A}^\sigma \in \mathbf{SH}(\mathbf{S}_\top^+)$. Finally, since \mathbf{A} is a subalgebra of \mathbf{A}^σ , $\mathbf{A} \in \mathbf{V}(\mathbf{S}_\top^+)$ as desired.

42. LOWER-BOUNDED SEMILATTICES

Let \mathbf{S}_\perp denote the variety of lower-bounded semilattices. Thus the members of \mathbf{S}_\perp have a new constant symbol, \perp and satisfy the identity $\perp \cdot x = \perp$. As in ¶41, \perp denotes the complex nullary operation obtained from \perp .

THEOREM 17. *If Δ is an equational base for \mathbf{BSI} , then*

$$\Delta_\perp = \Delta \cup \{ \perp \cdot 1 = \perp, (x \wedge \perp) \cdot (y \wedge \perp) = x \wedge y \wedge \perp \}$$

is an equational base for $\mathbf{V}(\mathbf{S}_\perp^+)$.

PROOF: One easily checks that $\mathbf{S}_\perp^+ \models \Delta_\perp$. For the converse, let \mathbf{A}_\perp be a subdirectly irreducible model of Δ_\perp . We wish to show that $\mathbf{A}_\perp \in \mathbf{V}(\mathbf{S}_\perp^+)$.

Let \mathbf{A} be the reduct of \mathbf{A}_\perp back to the language of Boolean groupoids. Then $\mathbf{A} \models \Delta$, so by assumption, $\mathbf{A} \in \mathbf{BSI}$. Furthermore, \mathbf{A} has an element b such that

$$(3) \quad \mathbf{A} \models b \cdot 1 = b \ \& \ (\forall x, y) ((x \wedge b) \cdot (y \wedge b) = x \wedge y \wedge b).$$

As in Theorem 16 we now switch to the canonical extension \mathbf{A}^σ . By ¶30, \mathbf{A}^σ is subdirectly irreducible. Since the terms involved in formula (3) are all positive, those conditions holds in \mathbf{A}^σ as well.

Claim: b is an atom and $\langle b \rangle$ is the monolith of \mathbf{A}^σ .

PROOF: Since $b \cdot 1 = b$, $\langle b \rangle$ is a congruence ideal of \mathbf{A}^σ . Suppose $0 < x < b$. Let $y = b - x$, so $0 < y < b$. By formula (3), $x \cdot y = x \wedge y \wedge b = 0$, contradicting the subdirect irreducibility of \mathbf{A}^σ . Therefore b is an atom, and the claim about the monolith follows immediately.

Again, following the argument used in Theorem 16, there is a semilattice \mathbf{S} , a subalgebra \mathbf{B} of \mathbf{S}^+ , and a surjective homomorphism $k: \mathbf{B} \twoheadrightarrow \mathbf{A}$. Applying duality, there is a $\mathbf{T} \in \mathbf{S}$ and a bounded homomorphism $h: \mathbf{T} \rightarrow \mathbf{B}_\delta$, and \mathbf{A}_δ is an inner substructure of \mathbf{B}_δ (see the diagram in article 41). Let $E = h^{-1}(b)$.

Claim: E is a downset of \mathbf{T} .

PROOF: Suppose $x \in E$ and $y \preceq x$ in \mathbf{T} . Then $x \cdot y = y$, so $(x, y, y) \in R_{\mathbf{T}}$. Applying the bounded morphism $h: (b, hy, hy) \in R_{\mathbf{B}_\delta}$. Therefore, in the complex algebra $(\mathbf{B}_\delta)^+ = \mathbf{B}^\sigma$, $b \cdot hy \geq hy$. But $b = b \cdot 1 \geq b \cdot hy$ and both b and hy are atoms of \mathbf{B}^σ , so it follows that $b = hy$, and therefore, $y \in E$.

Define ψ to be the equivalence relation on \mathbf{T} that identifies all elements of E and nothing else. One easily checks that ψ is a semilattice *congruence* relation on \mathbf{T} (the “Rees congruence”). Let $\hat{\mathbf{T}} = \mathbf{T}/\psi$. Then $\hat{\mathbf{T}}$ is a semilattice with a smallest element, E/ψ . Since $\psi \subseteq \ker h$, h induces a map $\hat{h}: \hat{\mathbf{T}} \rightarrow \mathbf{B}_\delta$, by $\hat{h}(x/\psi) = h(x)$. It is straightforward to verify that \hat{h} is a bounded morphism.

Finally, we make $\hat{\mathbf{T}}$ into a member of \mathbf{S}_\perp by taking $\perp = E/\psi$ on $\hat{\mathbf{T}}$. Similarly, \mathbf{B}_δ and \mathbf{A}_δ can be made into relational structures of type $\langle 3, 1 \rangle$ by defining \perp to be $\{b\}$, in both cases. \hat{h} is still a bounded morphism in this new category. Putting all of this together, and applying duality, each of \mathbf{T}^+ , $\mathbf{B}_\delta^+ = \mathbf{B}^\sigma$, and $\mathbf{A}_\delta^+ = \mathbf{A}^\sigma$ are members of $\mathbf{V}(\mathbf{S}_\perp^+)$. Since \mathbf{A}_\perp is a subalgebra of \mathbf{A}^σ (in the language containing the additional constant), $\mathbf{A}_\perp \in \mathbf{V}(\mathbf{S}_\perp^+)$.

The bottom of the lattice of subvarieties

43. SMALL BOOLEAN SEMILATTICES

There is only one Boolean semilattice of order 2, namely, 1^+ . (We will let \mathbf{n} denote an n -element chain, viewed as a semilattice.) Furthermore, $\mathbf{V}(1^+)$ is axiomatized, relative to Σ , by $x \cdot y = x \wedge y$. Also by $\downarrow x = x$.

PROOF: The only integral algebra satisfying $x \cdot y = x \wedge y$ is 1^+ . Since every subdirectly irreducible Σ -algebra is integral, that identity must define the variety.

There are six algebras of order 4. They all have the same Boolean structure. Let the atoms be denoted by a and b . Then the complex operation can be defined on

the atoms as follows:

| | $a \cdot a$ | $b \cdot b$ | $a \cdot b$ |
|------------------|-------------|-------------|-------------|
| $1^+ \times 1^+$ | a | b | 0 |
| 2^+ | a | b | a |
| A_1 | a | b | 1 |
| A_2 | a | 1 | a |
| A_3 | a | 1 | 1 |
| A_4 | 1 | 1 | 1 |

A_2 is a subalgebra of S^+ , where S is the three-element semilattice that is not a chain. The remaining three algebras are not finitely representable (by Corollary 13). However, it can be shown that all three are subalgebras of infinite complex algebras.

Suppose B is a 4-element algebra whose closure-reduct is simple and monadic. Then $a \cdot 1 = b \cdot 1 = 1$ and therefore $a \cdot b \geq (a \cdot 1) \wedge (b \cdot 1) \wedge (a \vee b) = 1$. So $B \cong A_i$ for $i = 1, 3$ or 4 .

Problem 4. Is A_i finitely based, for $i = 1, 3, 4$?

44. SIMPLE MONADIC ALGEBRAS

Problem 5. Is the 4-element monadic algebra finitely presented? (I'm not sure what this question means—there are a lot of coffee stains on the paper. The algebras A_i for $i = 1, 3, 4$ of article 43 all would seem to be monadic.)

45. THE BOTTOM OF THE LATTICE OF SUBVARIETIES OF BSI

The lattice of subvarieties of BSI has a single atom, since every nontrivial algebra contains the subalgebra $\{0, 1\}$, which is isomorphic to 1^+ . What are the covers of that variety? Are there only finitely many? Are they all finitely generated?

Problem 6. Let $V \models \Sigma$ and suppose that V covers $V(1^+)$ in the lattice of varieties. Is $V \subseteq \text{BSI}$?

46. 8 ELEMENT ALGEBRAS

Does every 8-element member of $\text{Mod}(\Sigma)$ contain a 4-element subalgebra? (More likely, we should be restricting to subdirectly irreducibles.) Does the variety generated by an 8-element algebra contain a 4-element subdirectly irreducible algebra?

47. MORE ON THE FOUR ELEMENT SIMPLE MONADIC ALGEBRAS

Let A be one of the algebras A_i for $i = 1, 3, 4$ of ¶43. Then $A \notin \text{HS}(S_{\text{fin}}^+)$.

PROOF: Suppose $A \in \text{HS}(S^+)$ for some semilattice S . By CEP, $\text{HS}(S^+) = \text{SH}(S^+)$ so A is a subalgebra of an algebra C^+ , where C is an upset of S . By the remarks at the end of ¶34, C , hence S must be infinite.

THEOREM 18. $V(S_{\text{fin}}^+) \subset \text{BSI}$.

PROOF: The algebra \mathbf{A} above is simple and finite, so it is splitting. Let BSI/\mathbf{A} denote the conjugate variety. Then from the above arguments, $S_{\text{fin}}^+ \subset \text{BSI}/\mathbf{A}$. The result follows.

Other identities

48. THE DISCRIMINATOR SUBVARIETY

Let \mathbf{D} be the subvariety of $\text{Mod } \Sigma$ defined by the identity

$$(4) \quad (x \cdot 1)' \cdot 1 = (x \cdot 1)'.$$

\mathbf{D} is a discriminator variety,⁴ in fact, the largest discriminator subvariety of $\text{Mod}(\Sigma)$. A discriminator term for this variety is

$$t(x, y, z) := x \wedge \downarrow(x \oplus y) \vee z \wedge (\downarrow(x \oplus y))'.$$

PROOF: Suppose that \mathbf{B} is a subdirectly irreducible member of \mathbf{D} . Let a be a nonzero member of B . Then by 8, $(\downarrow a] = J$ is a congruence ideal of \mathbf{B} . Recall that $\downarrow a$ is shorthand for $a \cdot 1$. By our assumption, $(a \cdot 1)'$ is closed, therefore $I = ((a \cdot 1)']$ is a congruence ideal. Since $J \cap I = \{0\}$, and $J \neq \{0\}$, we have, by the subdirect irreducibility of \mathbf{B} , that $(a \cdot 1)' = 0$, so $a \cdot 1 = 1$. Since a was an arbitrary nonzero element of B , it follows from Theorem 10 that \mathbf{B} is simple. It is easy to see that the term t induces a discriminator operation on every simple model of Σ . Thus this variety is a discriminator variety.

Conversely, let \mathbf{D}' be any discriminator subvariety of $\text{Mod}(\Sigma)$. Then every subdirectly irreducible member of \mathbf{D}' is simple. Applying Theorem 10 again, every simple algebra satisfies identity (4). Therefore, \mathbf{D}' satisfies this identity.

Equivalent forms for identity (4):

$$\begin{aligned} x &\leq (\downarrow x)^\circ \\ \downarrow x &\leq (\downarrow x)^\circ \end{aligned}$$

Notice that this variety does not contain an algebra \mathbf{S}^+ for any nontrivial semilattice \mathbf{S} . (The identity (4) requires that the complement of a closed element be closed. But in a semilattice, the complement of a downset is never a downset.)

On the other hand, this variety is nontrivial, since it contains every simple algebra of $\text{Mod}(\Sigma)$. In particular, the algebras \mathbf{A}_1 , \mathbf{A}_3 and \mathbf{A}_4 of ¶43 are all discriminator algebras.

Problem 7. Jipsen proved that every variety of residuated complex algebras is a discriminator variety. Can we get some kind of residuation in the above variety?

⁴This ought to be credited to Jipsen

49. MORE ON THE DISCRIMINATOR VARIETY D

Let D denote the largest discriminator subvariety of $\text{Mod}(\Sigma)$ (see article 48). Then $D \models \downarrow x \leq (\downarrow x)^\circ$. It follows that $S^+ \cap D = \{1^+\}$.

Problem 8. Is D a complex variety? Is $D \subseteq \text{BSI}$?

50. CONDITIONS EQUIVALENT TO LINEARITY

THEOREM 19. *In any model of Σ , the following are equivalent*

- (1) $x \cdot y = (x \cdot 1) \wedge (y \cdot 1) \wedge (x \vee y)$;
- (2) $x \cdot x = x$;
- (3) $x \cdot y \leq x \vee y$.

PROOF: (1) \Rightarrow (2): Taking $x = y$, $x \cdot x = (x \cdot 1) \wedge x = x$.

(2) \Rightarrow (3): By (2) and additivity:

$$x \vee y = (x \vee y)^2 = x^2 \vee (x \cdot y) \vee y^2 = x \vee (x \cdot y) \vee y.$$

(3) follows easily.

(3) \Rightarrow (1): Follows easily from (3) and monotonicity.

51. LINEAR SEMILATTICES

Let S be a semilattice. If $X \subseteq S$, then X is a subsemilattice if and only if $X \cdot X = X$. And S is linearly ordered if and only if every nonempty subset forms a subsemilattice. Combining these two observations, a semilattice S is linear if and only (if S) $x \cdot x = x$. Note that this is condition 2 of Theorem 19.

Let L denote the class of linearly-ordered semilattices. By the above observations, L^+ satisfies each of the three equivalent conditions of Theorem 19.

THEOREM 20. $V(L^+)$ is axiomatized by $\Sigma \cup \{x \cdot x = x\}$. In particular, it is a finitely based variety.

PROOF: Let B be a Boolean groupoid satisfying $\Sigma \cup \{x \cdot x = x\}$. Then B satisfies all three conditions of Theorem 19. By condition 1, the structure of B is entirely determined by that of B^\downarrow . But B^\downarrow lies in the variety $S_{4,3}$ of modal algebras.⁵ This variety of closure algebras is generated by the complex algebras of linearly ordered sets. It follows from our observation about condition 1, that the corresponding variety of Boolean groupoids is generated by linearly ordered semilattices.

⁵Wim, can you give me a short proof of this?

52. UP-DIRECTED SEMILATTICES

THEOREM 21. *Let $\mathbf{S} \in \mathbf{S}$. Then \mathbf{S} is up-directed if and only if*

$$(\delta) \quad \mathbf{S}^+ \models \downarrow(x^\circ) \wedge \downarrow(y^\circ) \leq \downarrow((x \wedge y)^\circ).$$

PROOF: First let \mathbf{S} be up-directed. Suppose that $X, Y \in \mathbf{S}^+$ and $z \in \downarrow(X^\circ) \cap \downarrow(Y^\circ)$. Then there are $x \in X^\circ$ and $y \in Y^\circ$ with $z \preceq x, y$. By up-directedness, there is $u \in \mathbf{S}$ such that $x, y \preceq u$. Since X° and Y° are upsets, $u \in X^\circ \cap Y^\circ$. Therefore $z \in \downarrow(X^\circ \cap Y^\circ) = \downarrow(X \cap Y)^\circ$ by article 27.

Conversely, suppose \mathbf{S}^+ satisfies the identity, and let $x, y \in \mathbf{S}$. Let $X = \uparrow x$ and $Y = \uparrow y$. Then $X = X^\circ$ and $Y = Y^\circ$. Since \mathbf{S} is a semilattice, the element $z = x \cdot y \in \mathbf{S}$, and $z \in \downarrow X \cap \downarrow Y \subseteq \downarrow((X \cap Y)^\circ)$. Therefore, $X \cap Y$ is nonempty. Let $u \in X \cap Y$. By the definition of X and Y , $x, y \preceq u$. This shows that \mathbf{S} is up-directed.

53. ANOTHER IDENTITY

Let ϵ be the identity

$$(\epsilon) \quad x \wedge (x' \cdot 1) \leq (x \cdot 1)' \cdot 1.$$

Note that the right-hand side can be rewritten as $x'^\circ \cdot 1$.

PROPOSITION 22. $\mathbf{3}^+ \not\models \epsilon$.

PROOF: Let $\mathbf{3} = \langle a \prec b \prec c \rangle$. Take $X = \{a, c\}$. Then $X' \cdot 1 = \downarrow b = \{a, b\}$, so $a \in \bar{X} \cap (X' \cdot 1)$. But $X \cdot 1 = 3$, so $(X \cdot 1)' \cdot 1 = \emptyset$.

Now let \mathbf{S} be the three-element semilattice that is not a chain, and \mathbf{T} be the semilattice $\mathbf{2} \times \mathbf{2}$. Then $\mathbf{T}^+ \not\models \epsilon$ (since $\mathbf{3}$ is a bounded homomorphic image of \mathbf{T}^+), but \mathbf{T}^+ does satisfy the identity (δ) of ¶52 since it is up-directed. On the other hand, $\mathbf{S}^+ \models \epsilon$, but \mathbf{S}^+ does not satisfy the up-directed identity.

Problem 9. Is $\mathbf{3}$ splitting (with conjugate identity ϵ) in \mathbf{BSI} ?

54. THE TOTAL RELATION

Let $\mathbf{K} = \{ \langle S, S^3 \rangle : S \text{ a set} \}$. What can we say about \mathbf{K}^+ ? It clearly satisfies Σ . Did we prove that it lies in \mathbf{BSI} ?

One easily checks that no member of \mathbf{K} has an inner substructure (or put another way, using Theorem 10, every member of \mathbf{K}^+ is simple), and \mathbf{K} is closed under bounded morphic images.

The variety $\mathbf{V}(\mathbf{K}^+)$ can be axiomatized (relative to what?) by $\downarrow x = (\downarrow x)^\circ$ and $\downarrow(x \cdot y) = x \cdot y$.⁶ Will one of these suffice? Can you substitute $x^2 = \downarrow x$?

Problem 10. Is this variety generated by its finite members? Is it locally finite? What are its subvarieties? What is its relationship to monadic algebras?

⁶Proof?

The algebra B_ω^+

55. THE FREE SEMILATTICE AS A DIRECT LIMIT

Let S denote the free (meet) semilattice on countably many generators, and S_n the free semilattice on n generators. Note that every S_n is an inner substructure (i.e., an upset) of S , and that $S = \bigcup_{n \in \omega} S_n$.

On the other hand, $S = \bigcup S_n$ is a bounded homomorphic image of $\bigcup S_n$ in a natural way. Applying duality, we conclude that

$$V(S^+) = \bigvee_{n \in \omega} V(S_n^+).$$

In particular, $V(S^+)$ is generated by its finite members. Surely, there is something more general going on here.

56. A FINITENESS PROPERTY OF B_ω^+

Let B_ω denote the infinite binary tree, and let B_n be a finite binary tree of height n . B_n is a BHI of B_ω . To see this, let θ_n be the equivalence relation defined on B_ω that identifies all of $\uparrow x$, for every x in B_ω of height n . It is easy to see that θ_n satisfies the conditions of ¶7, and $B_\omega/\theta_n \cong B_n$.

Applying duality, we have a chain of inclusions:

$$B_0^+ \hookrightarrow B_1^+ \hookrightarrow B_2^+ \hookrightarrow \dots B_\omega^+$$

For contrast, consider the following. Think of B_ω as the set of finite binary sequences, ordered by extension. Let U be the subset of B_ω consisting of those sequences that end in a 0, and let V be the complement of U . Let \mathfrak{F} be the set of finite subsemilattices of B_ω .

Note that $U \cdot V = B_\omega$, and therefore,

$$(\forall F \in \mathfrak{F}) \quad (U \cdot V) \cap F \neq (U \cap F) \cdot (V \cap F).$$

This means that the obvious approach to embedding B_ω^+ into a nonprincipal ultra-product of $\{F^+ : F \in \mathfrak{F}\}$ is not going to work. (Big deal!)

57. ON A_3 , A_4 AND B_ω^+ .

PROPOSITION 23. $A_3 \notin S(B_\omega^+)$ (See article 43).

PROOF: Suppose it were. Let X and Y denote the two atoms of A_3 , viewed as complexes of B_ω . Then $X \cdot X = X$ and $X \cdot Y = Y \cdot Y = B_\omega$. Choose a minimal element, y , of Y . Since $Y \subseteq X \cdot 1$ there is a minimal element x of X with $x \succeq y$. Say x is on the left subtree above y . Let z be the right-hand successor of y . Since $X \cdot X = X$, $z \in Y$, in fact, $\uparrow z \subseteq Y$. But $X \cdot 1 = 1$ implies that z must lie below some element of X , which is a contradiction.

On the other hand, let A be the subalgebra of B_ω^+ generated by $\{U, V\}$, where U and V are the complexes defined in the last two paragraphs of ¶56. Since $U \cdot U = U \cdot V = V \cdot V = B_\omega$ $A \cong A_4$, so $A_4 \in S(B_\omega^+)$.

58. MORE ON \mathbf{B}_ω

Let \mathbf{U} be an upset of \mathbf{B}_ω . Then U has minimal elements $\{u_1, u_2, \dots\}$. Therefore $U = \bigcup \uparrow u_i$. Each $\uparrow u_i \cong \mathbf{B}_\omega$. Thus \mathbf{U} is isomorphic to $\mathbf{B}_\omega^{(n)}$ or to $\mathbf{B}_\omega^{(\omega)}$. By duality, $\mathbf{U}^+ \in \mathbf{P}(\mathbf{B}_\omega^+)$. Similar statements holds for the upsets of \mathbf{B}_n .

PROPOSITION 24. *Let \mathbf{T} be the ternary tree of height 1. \mathbf{T} is not a bounded homomorphic image of \mathbf{B}_ω .*

PROOF: Suppose it were. Then there is a bounded equivalence relation θ on B_ω such that $\mathbf{T} \cong \mathbf{B}_\omega/\theta$. Since \mathbf{T} is a semilattice, θ is also a congruence relation on \mathbf{B}_ω . θ will have four equivalence classes, and each of them will have a minimum element, since \mathbf{T} is idempotent. Let the four minimal elements be a, b, c, \perp , where \perp is the minimum element of \mathbf{B}_ω .

Now $a/\theta \cdot b/\theta = \perp/\theta$. It follows from the boundedness of θ and the minimality of a and b that $a \cdot b = \perp$. But similarly, $a \cdot c = b \cdot c = \perp$ which is impossible.

THEOREM 25. $\mathbf{T}^+ \notin \mathbf{SP}_u(\mathbf{B}_\omega^+)$ and $\mathbf{T}^+ \notin \mathbf{V}\{\mathbf{B}_n^+ : n < \omega\}$.

PROOF: By Proposition 24 and duality, $\mathbf{T}^+ \notin \mathbf{S}(\mathbf{B}_\omega^+)$. Since \mathbf{T}^+ is finite, there is a first order sentence that says "I have a subalgebra isomorphic to \mathbf{T}^+ ". Therefore, $\mathbf{T}^+ \notin \mathbf{SP}_u(\mathbf{B}_\omega^+)$.

Now, for any n , \mathbf{B}_n is a bounded homomorphic image of \mathbf{B}_ω (see ¶56). It follows from the Proposition that \mathbf{T} is not a bounded homomorphic image of \mathbf{B}_n for any n , and therefore \mathbf{T}^+ can not be embedded into \mathbf{B}_n^+ . Since \mathbf{T}^+ is finite and subdirectly irreducible, it is splitting. Therefore, $\{\mathbf{B}_n^+ : n \in \omega\} \subseteq \mathbf{BSI}/\mathbf{T}^+$. From this, the second claim follows.

We can apply the results of ¶37 to $\mathbf{K} = \{\mathbf{B}_\omega^+\}$. $\mathbf{H}_\omega(\mathbf{B}_\omega^+) \subseteq \mathbf{P}(\mathbf{B}_\omega^+)$. Therefore $\mathbf{V}(\mathbf{B}_\omega^+)_{\text{fsi}} \subseteq \mathbf{SH}_\omega(\mathbf{B}_\omega^+) \subseteq \mathbf{SP}(\mathbf{B}_\omega^+)$, and therefore $\mathbf{V}(\mathbf{B}_\omega^+)_{\text{fsi}} \subseteq \mathbf{S}(\mathbf{B}_\omega^+)$. Since \mathbf{T}^+ is (finitely) subdirectly irreducible, we conclude from the above Theorem that $\mathbf{T}^+ \notin \mathbf{V}(\mathbf{B}_\omega^+)$.

Finally, in ¶56, we showed that \mathbf{A}_4 is embeddable in \mathbf{B}_ω^+ . Since \mathbf{A}_4 is not finitely representable, we conclude

$$\mathbf{V}\{\mathbf{B}_n^+ : n \in \omega\} \subset \mathbf{V}(\mathbf{B}_\omega^+) \subset \mathbf{BSI}/\mathbf{T}^+.$$

Problem 11. Is $\mathbf{V}(\mathbf{B}_\omega^+)$ generated by its finite members?

59. ONE MORE FACT ABOUT \mathbf{B}_ω^+

$\mathbf{B}_\omega^+ \models x^2 = x^3$.

PROOF: Let $X \subseteq B_\omega$. Think of B_ω as finite binary sequences. Suppose $a, b, c \in X$ and let $u = a \cdot b \cdot c \in X^3$. If any two of a, b and c are comparable, then $u \in X^2$. So we may assume that a, b, c are pairwise incomparable. As a finite sequence, $u = \langle u_0, u_1, \dots, u_n \rangle$. Then the lengths of a, b and c must be greater than n . Without loss of generality, $a_{n+1} = b_{n+1} = 0$ and $c_{n+1} = 1$. But then $a \cdot c = u$.

4. OTHER OBSERVATIONS ON BOOLEAN ALGEBRAS WITH OPERATORS

Boolean semigroups and monoids

60. SEMIGROUPS

Let \mathbf{V} denote the class of all semigroups. Is $\mathbf{V}(\mathbf{V}^+)$ finitely axiomatizable? (It is not decidable.)⁷

61. CLOSURE OPERATION ON A BOOLEAN MONOID

Let $\langle M, \cdot, e \rangle$ be a monoid. Then the operation ' \downarrow ' on M^+ defined by $\downarrow X = X \cdot 1$ is a closure operation. Of course we can no longer consider $\downarrow X$ to be a downset. However, it is, in the terminology of semigroup theory, the right ideal generated by X .

It may be better to use the term $1 \cdot x \cdot 1$ for a closure operation on a monoid. Pam Reich has shown that the term $x \vee x \cdot 1 \vee 1 \cdot x \vee 1 \cdot x \cdot 1$ is a closure operation on the complex algebra of any semigroup. The closed subsets are precisely the ideals.

62. WHICH BOOLEAN MONOIDS ARE BOOLEAN SEMILATTICES?

Let M be a monoid. Then $M^+ \models x \leq x^2$ if and only if M is a semilattice. Can this be used to obtain an axiomatization of Boolean semilattices from Boolean monoids?

63. FINITE MONOIDS

Let $M = \langle M, +, e \rangle$ be a finite monoid. Then M^+ satisfies the quasiidentity

$$(5) \quad x + x \leq x \ \& \ e \leq x + 1 \rightarrow e \leq x.$$

PROOF: Suppose $A \subseteq M$, $A + A \subseteq A$ and $e \in A + M$. Then there are $a \in A$ and $b \in M$ such that $e = a + b$. Therefore $2a + 2b = a + (a + b) + b = a + e + b = a + b = e$. Arguing inductively, $ka + kb = e$ for every positive integer k . Now the set $\{a, 2a, 3a, \dots\}$ is contained in A and is finite. Therefore there are positive integers $k < n$ such that $ka = na$. So by the above computation, $e = ka + kb = na + kb = (n - k)a + ka + kb = (n - k)a \in A$.

Note that $\langle \mathbb{Z}, +, 0 \rangle$ fails to satisfy formula (5), since $A = \{1, 2, 3, \dots\}$ is a witness. Consider also the following frame \mathbf{T} :

| $+$ | a | b | e |
|-----|-----|-----|-----|
| a | a | 1 | a |
| b | 1 | b | b |
| e | a | b | e |

⁷Are we sure of this? I can't remember the details.

\mathbf{T}^+ also fails to satisfy the quasiidentity. There is a bounded morphism $f: \mathbb{Q} \rightarrow \mathbf{T}$ given by

$$f(x) = \begin{cases} a & \text{if } x > 0 \\ e & \text{if } x = 0 \\ b & \text{if } x < 0. \end{cases}$$

Here is another quasiidentity that has the same properties. For some reason, we liked it better.

$$(\forall a, p, q) \ a \geq 2a \ \& \ p \leq a \ \& \ p + q = e \rightarrow e \leq a.$$

64. THE VARIETY OF BOOLEAN MONOIDS IS NOT GENERATED BY THE FINITE MONOIDS

In EDPC I, Blok and Pigozzi proved that (assuming equationally definable principal congruences) if χ is a quasiidentity and $\mathbf{Q} = \text{Mod}(\chi)$, then \mathbf{Q} has a largest subvariety, namely,

$$\mathbf{V}_{\mathbf{Q}} = \{ \mathbf{A} \in \mathbf{Q} : \mathbf{H}_{\omega}(\mathbf{A}) \subseteq \mathbf{Q} \}.$$

Here $\mathbf{H}_{\omega}(\mathbf{A}) = \{ \mathbf{A}/\theta : \theta \text{ is a compact congruence} \}$.

Let \mathbf{M} denote the class of monoids, and \mathbf{Q} the subquasivariety of $\mathbf{V}(\mathbf{M}^+)$ defined by the quasiidentity (5) of ¶63. ($\mathbf{V}(\mathbf{M}^+)$ has EDPC by the same argument used in ¶36.) Then $\mathbf{M}_{\text{fin}}^+ \subseteq \mathbf{Q} \subseteq \mathbf{V}(\mathbf{M}^+)$, and therefore $\mathbf{M}_{\text{fin}}^+ \subseteq \mathbf{V}_{\mathbf{Q}}$, since $\mathbf{M}_{\text{fin}}^+$ is closed under \mathbf{H}_{ω} . It follows that $\mathbf{V}(\mathbf{M}_{\text{fin}}^+) \subseteq \mathbf{V}_{\mathbf{Q}}$. A similar argument works for commutative Boolean monoids.

We have proved the following Theorem.

THEOREM 26. *Neither the variety of Boolean monoids nor the variety of commutative Boolean monoids is generated by the complex algebras of finite monoids.*⁸

Problem 12. Is the variety \mathbf{BSI} generated by its finite members?

65. A QUESTION FOR PAM

Let X be a set. Is $\langle \mathcal{P}(X^2), |, \Delta_X \rangle \in \mathbf{V}(\mathbf{M}^+)$, where \mathbf{M} is the class of monoids?

66. ANOTHER IDENTITY

What can you say about the identity $x \cdot y = x \cdot y \cdot 1$? (See article 54.)

67. YET ANOTHER IDENTITY

Problem 13. Consider the variety of associative Boolean groupoids that satisfy $x^2 = x^3$.

⁸In my notes, I had something more general: these varieties are not generated by their finite members. Can we obtain that result from this argument?

Other structures

68. MONO-UNARY ALGEBRAS

Let \mathbf{V} be a variety of mono-unary algebras. Then $\mathbf{V}(\mathbf{V}^+)$ is finitely axiomatizable and is identical to $\mathbf{V}((\mathbf{V}_{\text{fin}})^+)$.⁹

69. COMPLEX ALGEBRAS OF GROUPOIDS

Let \mathbf{G} denote the class of all groupoids. Then $\mathbf{V}(\mathbf{G}^+)$ is the variety of all Boolean groupoids.

Problem 14. Is $\mathbf{V}((\mathbf{G}_{\text{fin}})^+)$ equal to the variety of all Boolean groupoids?

Analogous questions can be asked for the class of all commutative groupoids and idempotent groupoids.

70. THE “SUPER IDEMPOTENT LAW”

Consider the variety of Boolean groupoids that satisfy only the identity $x \cdot x \geq x$. How much of the theory of Boolean semilattices continues to hold. (What about the associative law?)

71. BOOLEAN SEMILATTICES WITH RESIDUATION

Let $\langle S, \cdot \rangle$ be a semilattice. Form the structure $\langle \mathcal{P}(S), \cdot, \rightarrow \rangle$, where

$$X \rightarrow Y = \{ z \in S : X \cdot \{z\} \subseteq Y \}:$$

Is the variety of all such algebras axiomatized by:

$$\begin{aligned} x \cdot y &= y \cdot x \\ (x \cdot y) \cdot z &= x \cdot (y \cdot z) \\ (x \cdot y) \rightarrow z &= x \rightarrow (y \rightarrow z)? \end{aligned}$$

What happens if we add \top to the type?

We have an example of this on \mathbb{N} , but I don't understand it anymore.

72. RESIDUATION IN \mathbf{L}

Look at the residuation operation in \mathbf{L} .

⁹Do we have a proof of this?