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Non-Axiomatizability of the Amalgamation Class of Modular Lattice Varieties

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Abstract. We prove that if $\mathscr{V}$ is the variety generated by a finite modular lattice, then Amal($\mathscr{V}$) is not an elementary class. We also consider the same question for the variety generated by N_5.

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Key words. Amalgamation class, bounded obstruction property, elementary class.

The amalgamation classes of various varieties of algebras have been studied by a number of people [4, 5, 8]; and the amalgamation class has even been characterized in a few special cases [2, 3, 6, 9]. For the most part, these characterizations have not been in terms of first-order sentences of the language. For example, they may assert the existence of special kinds of subdirect product representations. The one exception is [3], in which it is proved that the amalgamation class of a finitely generated discriminator variety of finite type is an elementary class. (Of course, any variety with the amalgamation property also has an amalgamation class which is elementary.)

In this paper we prove that if $\mathscr{V}$ is the variety generated by a finite modular lattice, then the amalgamation class of $\mathscr{V}$ is not elementary. On the one hand, it is not surprising that modular lattices fail to share a 'nice' property with discriminator algebras, since the latter have a far stronger structure theory. On the other hand, [2] contains characterizations of the amalgamation classes for both discriminator algebras and modular lattices that are remarkably similar.

The proof of the main theorem relies on results of Albert and Burris [1], relating the axiomatizability of the amalgamation class to what they call the bounded obstruction property. We begin by reviewing some relevant definitions and theorems. A general reference for concepts not defined here is [7].

The congruence lattice of an algebra $\mathbf{A}$ is denoted Con $\mathbf{A}$, with least and greatest elements 0 and 1, respectively. Let $\mathscr{V}$ be a variety of algebras. An algebra $\mathbf{A} \in \mathscr{V}$ is

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an *amalgamation base of* $\mathscr{V}$ if, for every $\mathbf{B}_0$, $\mathbf{B}_1 \in \mathscr{V}$ and embeddings f_i of $\mathbf{A}$ into $\mathbf{B}_i$ (for $i = 0, 1$), there is an algebra $\mathbf{C} \in \mathscr{V}$ and embeddings $g_i : \mathbf{B}_i \to \mathbf{C}$ ($i = 0, 1$) such that $g_0 \circ f_0 = g_1 \circ f_1$. In this case we say that the amalgam $\langle \mathbf{A}; \mathbf{B}_0, f_0, \mathbf{B}_1, f_1 \rangle$ can be amalgamated by $\langle \mathbf{C}; g_0, g_1 \rangle$. The *amalgamation class of* $\mathscr{V}$, denoted Amal($\mathscr{V}$), is the class of all amalgamation bases. $\mathscr{V}$ has the *amalgamation property* if and only if Amal($\mathscr{V}$) $= \mathscr{V}$.

The techniques used in [1] for characterizing Amal($\mathscr{V}$) involve studying the way certain amalgams fail to amalgamate. Suppose that $\langle \mathbf{A}; \mathbf{B}_0, f_0, \mathbf{B}_1, f_1 \rangle$ cannot be amalgamated. An *obstruction* is any subalgebra $\mathbf{B}'_1$ of $\mathbf{B}_1$ such that $\langle \mathbf{A}'; \mathbf{B}_0, f'_0, \mathbf{B}'_1, f'_1 \rangle$ cannot be amalgamated, where $A' = f_1^{-1}(B'_1)$ and $f'_i = f_i \restriction_{A'}$, for $i = 0, 1$.

DEFINITION. Let $\mathscr{V}$ be a locally finite variety. Amal($\mathscr{V}$) has the *bounded obstruction property with respect to* $\mathscr{V}$ if, for every $k \in \omega$, there exists $n \in \omega$ such that if $\langle \mathbf{A}; \mathbf{B}, f, \mathbf{C}, g \rangle$ cannot be amalgamated, $\mathbf{C} \in$ Amal($\mathscr{V}$) and $|B| < k$, then there is an obstruction $\mathbf{C}' \leqslant \mathbf{C}$ such that $|C'| < n$.

THEOREM 1. ([1, Corollary 2.6]) *Let* $\mathscr{V}$ *be a finitely generated variety of finite type. Then* Amal($\mathscr{V}$) *has the bounded obstruction property with respect to* $\mathscr{V}$ *if and only if* Amal($\mathscr{V}$) *is elementary.*

Remark: Theorem 1 actually holds for universal Horn classes, not just varieties.

THEOREM 2. *Let* $\mathscr{V}$ *be a finitely generated, nondistributive variety of modular lattices. Then* Amal($\mathscr{V}$) *is not elementary.*

Proof. We will use Theorem 1, in other words, we will show that such a variety has unbounded obstructions. So let $\mathbf{L}$ be a finite, modular, nondistributive lattice, and let $\mathscr{V} = \boldsymbol{HSP}(\mathbf{L})$, the variety generated by $\mathbf{L}$. By Jónsson's lemma, the subdirectly irreducible members of $\mathscr{V}$ are all members of $\boldsymbol{HSP}_{\mathbf{u}}(\mathbf{L}) = \boldsymbol{HS}(\mathbf{L})$, since L is finite. Therefore, every subdirectly irreducible lattice in $\mathscr{V}$ has cardinality at most $|L|$, and furthermore, will be simple. Let $\mathbf{M}$ be such a lattice of largest cardinality. Since $\mathbf{L}$ is nondistributive, $|M| \geqslant 5$.

Denote the least and greatest elements of $\mathbf{M}$ by z and u respectively, and let a be an atom of $\mathbf{M}$. Let $\mathbf{2}$ be the usual lattice on $\{0, 1\}$ and $\mathbf{B} = \mathbf{M} \times \mathbf{2}$. Define $f \colon \mathbf{2} \to \mathbf{B}$ by $f(0) = (z, 0)$ and $f(1) = (a, 1)$. Our plan is to construct a lattice $\mathbf{C} \in$ Amal($\mathscr{V}$) (claim 2.1) and, for every $n \in \omega$, a homomorphism $g_n \colon \mathbf{2} \to \mathbf{C}$ such that $\langle \mathbf{2}; \mathbf{B}, \mathrm{f}, \mathbf{C}, g_n \rangle$ cannot be amalgamated (claim 2.2), and every obstruction has cardinality at least n (claim 2.3). Then, by Theorem 1, Amal($\mathscr{V}$) will not be elementary.

We first define $\mathbf{C}$. For $x \in M$ and $n \in \omega$, define $x[n] \in M^{\omega}$ by

$$x[n]_j = \begin{cases} u, & \text{for } j < n, \\ x, & \text{for } j = n, \\ z, & \text{for } j > n, \end{cases} \quad \text{for } j \in \omega,$$

and let $C = \{x[n] : x \in M \text{ and } n \in \omega\}$.

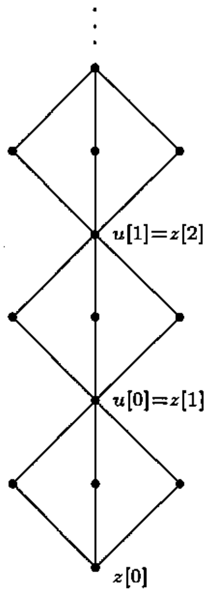

Fig. 1.

Some remarks about **C**:

- $z[n+1] = u[n]$, for all $n \in \omega$.
- **C** is the 'lexicographic product' of ω with $\mathbf{M} - \{u\}$, that is,

$$x[n] < y[m] \quad \Leftrightarrow \quad n < m \text{ or } (n = m \;\&\; x < y).$$

- **C** is a sublattice of $\mathbf{M}^\omega$. Computing in $\mathbf{M}^\omega$, $x[n] \vee y[m] = x[n]$ if $n > m$, $y[m]$ if $n < m$ or $(x \vee y)[n]$ if $n = m$. A similar computation holds for $x \wedge y$. **C** resembles infinitely many copies of **M** 'stacked up' so that the largest element of one copy is identified with the smallest element of the next. (See Figure 1 for a picture of **C** when $\mathbf{M} \cong \mathbf{M}_3$, the 5-element simple lattice.)
- For $x \notin \{z, u\}$, $x[n]$ is the unique element y of **C** such that $y_n = x$.
- **C** is a subdirect power of **M**.

Finally, for $n \in \omega$, we define $g_n : 2 \to \mathbf{C}$ by $g_n(0) = z[0]$ and $g_n(1) = u[n]$.

LEMMA 3. *Let α be a proper, strictly meet-irreducible congruence on* **C**. *Then there is $j \in \omega$ such that for all $x, y \in C$, $x \equiv y \pmod{\alpha} \Leftrightarrow x_j = y_j$. In particular,* $\mathbf{C}/\alpha \cong \mathbf{M}$.

Proof. From above, $\mathbf{C} \leqslant \mathbf{M}^\omega$. α is strictly meet-irreducible, so by Jónsson's lemma, there is an ultrafilter U on ω such that $\eta_U \restriction_C \subseteq \alpha$. (Here, η_U is the congruence on $\mathbf{M}^\omega$ induced by U, and $\eta_U \restriction_C$ is the restriction to C.) If U is nonprincipal, then for every $x[n] \in C$, $\{j \in \omega : x[n]_j = z\} = \omega - n \in U$, thus $x[n] \equiv z[0] \pmod{\alpha}$, implying that α is the congruence $\mathbf{1}_C$, contrary to our assumption.

Therefore U must be the principal ultrafilter generated by some $j \in \omega$. Since $\mathbf{C} \leqslant \mathbf{M}^{\omega}$ is subdirect, it follows that $\mathbf{C}/(\eta_U \restriction_C) \cong \mathbf{M}$ (the jth component). Since $\mathbf{C}/\alpha$ is a nontrivial homomorphic image of $\mathbf{C}/(\eta_U \restriction_C)$ and $\mathbf{C}/(\eta_U \restriction_C)$ is simple, we conclude that $\eta_U \restriction_C = \alpha$. Thus $x \equiv y \pmod{\alpha} \Leftrightarrow x_j = y_j$. In other words, α is the kernel of the projection of $\mathbf{C}$ onto $\mathbf{M}$ taking $x \mapsto x_j$. So $\mathbf{C}/\alpha \cong \mathbf{M}$. □

COROLLARY 4. *Every nontrivial homomorphic image of* **C** *is isomorphic to a subdirect power of* **M**.

Proof. If β is a congruence on **C**, then $\beta = \bigwedge_{i \in I} \alpha_i$, each α_i a strictly meet-irreducible congruence on **C**. By Lemma 3, $\mathbf{C}/\alpha_i \cong \mathbf{M}$. Therefore, we have the subdirect product representation $\mathbf{C}/\beta \hookrightarrow \Pi_{i \in I} \mathbf{C}/\alpha_i \cong \mathbf{M}^I$. □

In order to proceed with the proof of Thereom 2, we need one last theorem. Define $\mathscr{V}_{\mathrm{mi}}$ to be the set of all $\mathbf{N} \in \mathscr{V}_{\mathrm{si}}$ such that **N** has no proper subdirectly irreducible extension in $\mathscr{V}$.

THEOREM 5. ([**6**, Theorem 5.1]) *Let* $\mathscr{V}$ *be a finitely generated variety of lattices, and* $\mathbf{A} \in \mathscr{V}$. *Then* $\mathbf{A} \in \mathrm{Amal}(\mathscr{V})$ *iff for every* $\mathbf{D} \in \mathscr{V}$ *with* $\mathbf{A} \leqslant \mathbf{D}$ *and every* $\mathbf{N} \in \mathscr{V}_{\mathrm{mi}}$, *every homomorphism* $h\colon \mathbf{A} \to \mathbf{N}$ *extends to a homomorphism* $\bar{h}\colon \mathbf{D} \to \mathbf{N}$.

We can now prove the three claims that comprise the proof of Theorem 2.

CLAIM 2.1. $\mathbf{C} \in \mathrm{Amal}(\mathscr{V})$.

Proof. We will apply Theorem 5 to **C**. Let **D** be an extension of **C**, $\mathbf{N} \in \mathscr{V}_{\mathrm{mi}}$ and $h\colon \mathbf{C} \to \mathbf{N}$. If h is a trivial map, then it obviously extends to **D**. So assume h is nontrivial. Let $\alpha = \ker h$. By Corollary 4, every non-trivial homomorphic image of **C** is a subdirect power of **M**. On the other hand, $\mathbf{C}/\alpha$ is isomorphic to a subalgebra of **N**. Thus $|N| \geqslant |C/\alpha| \geqslant |M|$. But **M** was chosen to be of maximal cardinality in $\mathscr{V}_{\mathrm{si}}$, so we conclude that $|N| = |C/\alpha| = |M|$, from which it follows that $\mathbf{N} \cong \mathbf{C}/\alpha \cong \mathbf{M}$. In particular, α is a coatom of Con **C**.

Now by Birkhoff's theorem, **D** is isomorphic to a subdirect product of subdirectly irreducible lattices, say $\mathbf{D} \leqslant \Pi_{i \in I} \mathbf{D}_i$. By Jónsson's lemma, there is an ultrafilter U on I such that $\eta_U \restriction_C \leqslant \alpha$. Then we have an induced embedding of $\mathbf{C}/(\eta_U \restriction_C)$ into $\mathbf{D}/(\eta_U \restriction_D)$ and $\mathbf{C}/(\eta_U \restriction_C)$ maps onto $\mathbf{C}/\alpha \cong \mathbf{M}$ as well (Figure 2). This yields $|D/(\eta_U \restriction_D)| \geqslant |C/(\eta_U \restriction_C)| \geqslant |C/\alpha| = |M|$. Since each $\mathbf{D}_i$ is subdirectly irreducible (for $i \in I$), it lies in $\boldsymbol{HS}(\mathbf{L})$, which is, up to isomorphism, a finite set of finite lattices. Therefore $\mathbf{D}/(\eta_U \restriction_D)$, being an ultraproduct over that set, is isomorphic to one of its factors. In particular, $\mathbf{D}/(\eta_U \restriction_D)$ is subdirectly irreducible. Again, by the maximality of $|M|$, $\mathbf{D}/(\eta_U \restriction_D) \cong \mathbf{C}/(\eta_U \restriction_C) \cong \mathbf{C}/\alpha \cong \mathbf{M}$. Therefore, $\eta_U \restriction_C = \alpha$, and the desired extension of h can be constructed from the canonical map $\mathbf{D} \twoheadrightarrow \mathbf{D}/(\eta_U \restriction_D)$. Thus, by Theorem 5, $\mathbf{C} \in \mathrm{Amal}(\mathscr{V})$. □

CLAIM 2.2. *For every* $n \in \omega$, $\langle \mathbf{2}; \mathbf{B}, f, \mathbf{C}, g_n \rangle$ *cannot be amalgamated.*

Proof. Suppose $\langle \mathbf{D}; \bar{f}, \bar{g} \rangle$ amalgamates $\langle \mathbf{2}; \mathbf{B}, f, \mathbf{C}, g_n \rangle$ (Figure 3). Recall that a is an atom of **M**, and $\mathbf{B} = \mathbf{M} \times \mathbf{2}$. Let p be a map of **D** onto a subdirectly irreducible

$$\begin{tikzcd}
\mathbf{D} \arrow[r, two heads] & \mathbf{D}/(\eta_U \restriction_D) & \\
\mathbf{C} \arrow[u] \arrow[r, two heads] & \mathbf{C}/(\eta_U \restriction_C) \arrow[u] \arrow[r, two heads] & \mathbf{C}/\alpha \cong \mathbf{M}
\end{tikzcd}$$

Fig. 2.

lattice $\mathbf{D}'$, such that $p \circ \bar{f}(z, 0) \neq p \circ \bar{f}(a, 0)$. By the simplicity of $\mathbf{M}$ and $\mathbf{2}$, Con $\mathbf{B} = \{0, \eta_0, \eta_1, 1\}$ where

$$(x_0, x_1) \equiv (y_0, y_1) \pmod{\eta_j} \quad \Leftrightarrow \quad x_j = y_j, \text{ for } j = 0, 1.$$

Let $\alpha = \ker(p \circ \bar{f})$. Now $(z, 0) \not\equiv (a, 0) \pmod{\alpha}$ implies that $1_C \neq \alpha \neq \eta_1$. If $\alpha = 0$, then, since $\mathbf{B}/\alpha$ can be embedded into $\mathbf{D}'$, we have $|M| < |B| \leqslant |D'|$, contradicting the maximality of $\mathbf{M}$. Therefore $\alpha = \eta_0$, $\mathbf{B}/\alpha \cong \mathbf{M}$ and again, by maximality, the embedding of $\mathbf{B}/\alpha$ into $\mathbf{D}'$ is an isomorphism. So $\mathbf{D}' \cong \mathbf{M}$. In particular, since a is an atom of $\mathbf{M}$, $p \circ \bar{f}(a, 1)$ is an atom of $\mathbf{D}'$.

Observe that

$$(*) \quad p \circ \bar{g} \circ g_n(0) = p \circ \bar{f} \circ f(0) = p \circ \bar{f}(z, 0)$$
$$\neq p \circ \bar{f}(a, 1) = p \circ \bar{f} \circ f(1) = p \circ \bar{g} \circ g_n(1).$$

Therefore, $\bar{g}$ is nontrivial. By Corollary 4, $\mathbf{C}/\ker(p \circ \bar{g})$ is a subdirect power of $\mathbf{M}$, and can be embedded into $\mathbf{D}' \cong \mathbf{M}$. Therefore that induced embedding is also an isomorphism. Applying Lemma 3 to $\ker(p \circ \bar{g})$, there is $j \in \omega$ such that $p \circ \bar{g}(x) = p \circ \bar{g}(y) \Leftrightarrow x_j = y_j$. If $j > n$ then $p \circ \bar{g} \circ g_n(0) = p \circ \bar{g}(z[0]) = p \circ \bar{g}(u[n]) = p \circ \bar{g} \circ g_n(1)$, contradicting $(*)$. We conclude that $j \leqslant n$ and $p \circ \bar{g} \circ g_n(1) = p \circ \bar{g}(u[n])$

$$\begin{tikzcd}
& \mathbf{D}' & \\
& \mathbf{D} \arrow[u, "p"] & \\
\mathbf{B} \arrow[ur, "\bar{f}"] & & \mathbf{C} \arrow[ul, "\bar{g}"'] \\
& \mathbf{2} \arrow[ul, "f"] \arrow[ur, "g_n"'] &
\end{tikzcd}$$

Fig. 3.

will be the largest element of $\mathbf{D}'$ (since $u[n]_j = u$ is the largest element of $\mathbf{M}$). But by (*), $p \circ \bar{g} \circ g_n(1) = p \circ \bar{f}(a, 1)$ is an atom of $\mathbf{D}'$. This contradiction proves the claim. □

CLAIM 2.3. *Let* $\mathbf{A}$ *be an obstruction to* $\langle \mathbf{2}; \mathbf{B}, f, \mathbf{C}, g_n \rangle$, *and* $\mathbf{A} \leqslant \mathbf{C}$. *Then* $|A| > n$.

Proof. By the definition of obstruction, $\langle \mathbf{2}', \mathbf{B}, f\restriction_{2'}, \mathbf{A}, g_n\restriction_{2'} \rangle$ cannot be amalgamated, where $2' = g_n^{-1}(A)$. Thus $|2'| > 1$, in other words, $\mathbf{2}' = \mathbf{2}$, and $A \supseteq g_n(2) = \{z[0], u[n]\}$.

There is no map $r: \mathbf{A} \to \mathbf{2}$ such that $r \circ g_n$ is the identity on 2. For suppose such an r existed. Let s: $\mathbf{B} \to \mathbf{2}$ be defined by $s(x, y) = y$. Then one can easily check that $\langle \mathbf{A} \times \mathbf{B}; \bar{f}, \bar{g} \rangle$ amalgamates $\langle \mathbf{2}; \mathbf{B}, f, \mathbf{A}, g_n \rangle$, where $\bar{f}(x) = (g_n s(x), x)$ and $\bar{g}(x) = (x, fr(x))$.

For $k \in \omega$, Let $C_k = \{x \in C: z[k] < x < u[k]\}$. Observe that for $k \neq l$, C_k and C_l are disjoint. To show $|A| > n$ it suffices to show that for all $k \leqslant n$, $A \cap C_k \neq \emptyset$. Suppose to the contrary, that A is disjoint from C_k, and $k \leqslant n$. We define r: $\mathbf{A} \to \mathbf{2}$ by $r(x) = 0$ if $x \leqslant z[k]$ and 1 otherwise. Obviously r preserves joins. To check meets, observe that if $x, y \nleqslant z[k]$, then since $A \cap C_k = \emptyset$, $x, y \geqslant u[k]$, therefore $x \wedge y \geqslant u[k]$. Also, $r(z[0]) = 0$ and $r(u[n]) = 1$, so r is a one-sided inverse to g_n. This contradicts the assertion in the previous paragraph. Claim 2.3, and Theorem 2 are proved. □

Other Varieties of Lattices

It is natural to wonder whether the techniques used to prove Theorem 2 can be applied to any other varieties of lattices. Since Theorem 1 holds only for finitely generated varieties, we might consider the variety $\boldsymbol{HSP}(\mathrm{L})$, where $\mathbf{L}$ is a finite, nonmodular lattice. In general, our knowledge of the amalgamation bases in these varieties is limited. However, we do have a good understanding of Amal($\mathcal{N}$), where $\mathcal{N} = \boldsymbol{HSP}(\mathrm{N})$ and $\mathbf{N}$ is the lattice of Figure 4. So we consider this case in some detail.

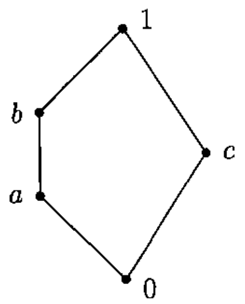

Fig. 4.

In [**6**, Theorem 6.1] Jónsson proved the following Theorem.

THEOREM 6. *A finite lattice* $\mathbf{L}$ *is a member of* Amal($\mathscr{N}$) *if and only if* $\mathbf{L}$ *is a subdirect power of* $\mathbf{N}$ *and the 3-element chain,* $\mathbf{3}$*, is not a homomorphic image of* $\mathbf{L}$.

Following the argument in Theorem 2, we define $\mathbf{M}$ to be the subdirectly irreducible lattice of largest cardinality. In other words, $\mathbf{M} \cong \mathbf{N}$. However, in this case, $\mathbf{M}$ is not simple. We can still construct the lattice $\mathbf{C}$, but Claim 2.1 fails: $\mathbf{C} \notin \mathrm{Amal}(\mathscr{N})$.

To see this, we apply Theorem 5. $\mathbf{C}$ is a sublattice of $\mathbf{N}^\omega$. There is a homomorphism $h: \mathbf{C} \to \mathbf{N}$ which maps the intervals $b[0]/0[0]$ to 0, $b[1]/c[0]$ to a and all remaining elements (i.e. the filter generated by $c[1]$) to b. Suppose there is a map $\bar{h}: \mathbf{N}^\omega \to \mathbf{N}$ extending h. Let $\theta = \ker \bar{h}$.

If $\bar{h}$ is surjective, then $\mathbf{N}^\omega/\theta \cong \mathbf{N}$, so θ is strictly meet-irreducible. Therefore there is an ultrafilter U on ω such that $\eta_U \subseteq \theta$. But arguing as before, $\mathbf{N}^\omega/\eta_U \cong \mathbf{N}$ implying that $\theta = \eta_U$. Now $h(b[0]) \neq h(c[0])$ implies that $\{j \in \omega : b[0]_j \neq c[0]_j\} = \{0\} \in U$. From this we deduce that $h(b[0]) \neq h(a[0])$, which is false.

Thus $\bar{h}$ is not surjective, so its image must be either $\{0, a, b\}$ or $\{0, a, b, 1\}$. We conclude that if $\mathbf{C} \in \mathrm{Amal}(\mathscr{N})$, then $\mathbf{N}^\omega$ has $\mathbf{3}$ for a homomorphic image. But this contradicts the following Theorem, since $\mathbf{3}$ is not a direct product of homomorphic images of $\mathbf{N}$.

THEOREM 7. *Let* $\mathscr{S}$ *be a finite set of finite algebras generating a congruence distributive variety. Every finite homomorphic image of a product of members of* $\mathscr{S}$ *is isomorphic to a product of homomorphic images of members of* $\mathscr{S}$*, i.e.* $\boldsymbol{HP}(\mathscr{S})_{\mathrm{fin}} \subseteq \boldsymbol{PH}(\mathscr{S})$.

Proof. Let $\mathbf{L} = \Pi_{i \in I} \mathbf{S}_i$, each $\mathbf{S}_i \in \mathscr{S}$. Let θ be a congruence on $\mathbf{L}$, and $\mathbf{L}/\theta$ finite. Then $\theta = \alpha_0 \wedge \alpha_1 \wedge \cdots \wedge \alpha_n$, each α_j a strictly meet-irreducible member of Con $\mathbf{L}$. By Jónsson's Lemma, for each $j \leqslant n$ there is an ultrafilter U_j on I such that the induced congruence $\eta_{U_j} \subseteq \alpha_j$. It may happen that for $k \neq j$, the ultrafilters U_j and U_k are equal. In that case, replace both α_j and α_k by the single congruence $\beta = \alpha_j \wedge \alpha_k$. By repeating this process, we may assume that there are congruences $\beta_0, \beta_1, \ldots, \beta_m$ on $\mathbf{L}$ and pairwise distinct ultrafilters $U_0, U_1, \ldots, U_m$ on I such that

$$\theta = \beta_0 \wedge \beta_1 \wedge \cdots \wedge \beta_m \quad \text{and} \quad \eta_{U_j} \subseteq \beta_j \quad \text{for } j = 0, 1, \ldots, m$$

(but the βs may not be meet-irreducible).

We claim that

$$(\ddagger) \quad \forall k \leqslant m \left[\bigwedge_{j \neq k} \eta_{U_j} \right] \circ \eta_{U_k} = 1_L.$$

For this, fix $k \leqslant m$ and pick $x, y \in L$. Let $V = \bigcap_{j \neq k} U_j$. Then V is a filter on I and $\eta_V = \bigcap_{j \neq k} \eta_{U_j}$. Since $U_0, U_1, \ldots, U_m$ are pairwise distinct ultrafilters, $V \nsubseteq U_k$. Let $E \in V - U_k$. Define $z \in L$ by

$$z_i = \begin{cases} x_i, & \text{for } i \in E, \\ y_i, & \text{for } i \in I - E. \end{cases}$$

Since U_k is an ultrafilter excluding E, $I-E \in U_k$. It follows that $x \eta_V z \eta_{U_k} y$, proving our claim.

Now since $\eta_{U_j} \subseteq \beta_j$, for all j, (‡) implies that $[\wedge_{j \neq k} \beta_j] \circ \beta_k = 1$ as well. Therefore $\mathbf{L}/\theta \cong \Pi_{j=0}^m \mathbf{L}/\beta_j$. $\mathbf{L}/\eta_{U_j}$ is an ultraproduct of members of $\mathscr{S}$. Since $\mathscr{S}$ is a finite set of finite algebras, such an ultraproduct must be isomorphic to some $\mathbf{S} \in \mathscr{S}$. Therefore, each $\mathbf{L}/\beta_j$ is a homomorphic image of a member of $\mathscr{S}$, as desired. □

Thus, in order to apply the line of reasoning used in Theorem 2, we will need a new construction for the lattice $\mathbf{C}$. The alert reader will have noticed that in Theorem 2, instead of a single 'uniform' (and infinite) lattice $\mathbf{C}$, we could have used a sequence of finite lattices (the quotients $u[n]/z[0]$, for $n = 1, 2, \ldots$) to prove the existence of unbounded obstructions. For the variety $\mathscr{N}$, we might look for finite lattices $\mathbf{C}_n$, each an amalgamation base of $\mathscr{N}$ (for this we can use Theorem 6) and containing an obstruction $\mathbf{C}'$ with $|C'| > n$.

Of course, there is no reason for $\mathbf{2}$ to play the role it does in Theorem 2. In fact the following Theorem suggests that $\mathbf{2}$ does not yield any unbounded obstructions at all. We use the notation $\mathbf{F}_{\mathscr{V}}(n)$ to denote the algebra in the variety $\mathscr{V}$ freely generated by an n element set.

THEOREM 8. *Let $m = |\mathbf{F}_{\mathscr{N}}(7)|$ and let $\langle \mathbf{2}; \mathbf{A}, f, \mathbf{C}, g \rangle$ fail to amalgamate in $\mathscr{N}$, where $\mathbf{C} \in \mathrm{Amal}(\mathscr{N})$ and C is finite. Then there is an obstruction $\mathbf{C}'$ such that $\mathbf{C}' \leqslant \mathbf{C}$ and $|C'| \leqslant m$.*

Proof. As $\mathscr{N}$ is generated by a finite lattice, it is locally finite. Therefore, m will indeed be a finite integer. By assumption $\langle \mathbf{2}; \mathbf{A}, f, \mathbf{C}, g \rangle$ cannot be amalgamated in $\mathscr{N}$. Now, because of the amalgamation failure, one of the following conditions fails to hold:

(1) for every distinct pair $x, y \in A$, there are $\mathbf{D} \in \mathscr{N}$ and $\bar{f}: \mathbf{A} \to \mathbf{D}$, $\bar{g}: \mathbf{C} \to \mathbf{D}$ such that $\bar{f} \circ f = \bar{g} \circ g$ and $\bar{f}(x) \neq \bar{f}(y)$;

(2) for every distinct pair $x, y \in C$, there are $\mathbf{D} \in \mathscr{N}$ and $\bar{f}: \mathbf{A} \to \mathbf{D}$, $\bar{g}: \mathbf{C} \to \mathbf{D}$ such that $\bar{f} \circ f = \bar{g} \circ g$ and $\bar{g}(x) \neq \bar{g}(y)$.

(See for example [**5**, Lemma 2].)

Suppose (2) fails. Let $x, y \in C$ witness that failure. There is a map $\bar{g}: \mathbf{C} \to \mathbf{N}$ such that $\bar{g}(x) \neq \bar{g}(y)$, since $\mathbf{N}$ is the unique maximal member of $\mathscr{N}_{\mathrm{si}}$. Therefore, there is no map $\bar{f}: \mathbf{A} \to \mathbf{N}$ such that $\bar{g} \circ g = \bar{f} \circ f$. Choose a subset S of C such that $\bar{g}(S) = \bar{g}(C)$ and $x, y \in S$. Clearly such an S can be chosen so that $|S| \leqslant |N| = 5$. Let $\mathbf{C}'$ be the subalgebra of $\mathbf{C}$ generated by $S \cup \{g(0), g(1)\}$. Then $\mathbf{C}'$ will be a homomorphic image of $\mathbf{F}_{\mathscr{N}}(7)$, so $|C'| \leqslant m$. Furthermore, $\mathbf{C}'$ is an obstruction since $\langle \mathbf{2}; \mathbf{A}, f, \mathbf{C}', g \rangle$ still violates condition (2).

Now assume that (1) fails. As before, there is a map $\bar{f}: \mathbf{A} \to \mathbf{N}$ such that for no $\bar{g}: \mathbf{C} \to \mathbf{N}$ does $\bar{f} \circ f = \bar{g} \circ g$. By Theorem 6, since $\mathbf{C}$ is a finite amalgamation base of $\mathscr{N}$, $\mathbf{C}$ has a subdirect power representation $\mathbf{C} \leqslant \mathbf{N}^n$ (some $n < \omega$) and $\mathbf{3} \notin H(\mathbf{C})$. Let

p_i denote the ith coordinate projection of $\mathbf{C}$ onto $\mathbf{N}$, and $\eta_i = \ker p_i$. Without loss of generality, we may assume that the subdirect representation is irredundant, that is, for every $j < n$, $\bigwedge_{i \neq j} \eta_i \neq 0_C$.

Suppose that for some $i < n$, the quotient $p_ig(1)\,p_ig(0)$ is neither trivial nor equal to the critical quotient b/a of $\mathbf{N}$. Then since $\bar{f}f(\mathbf{2})$ is a 2-element sublattice of $\mathbf{N}$, there is a map $q : \mathbf{N} \to \mathbf{N}$ such that $q \circ p_i \circ g = \bar{f} \circ f$. Setting $\bar{g} = q \circ p_i$ yields a contradiction.

Therefore, by renumbering the indices if necessary, $\mathbf{C} \leqslant \mathbf{N}^n$ and there is $l \leqslant n$ such that $p_ig(0) = a$ and $p_ig(1) = b$, for $i < l$, and $p_ig(0) = p_ig(1)$, for $l \leqslant i < n$. We will show that there is an obstruction $\mathbf{C}' \leqslant \mathbf{C}$ such that $\mathbf{C}' \cong \mathbf{N}$. This will certainly satisfy the requirements of the Theorem. For this it suffices to show that there is an element $e \in C$ such that $p_i(e) = c$, for $i < 1$. For then $\mathbf{C}'$ will be the subalgebra of $\mathbf{C}$ generated by $\{g(0), g(1), e\}$.

In order to make the proof more intuitive, we shall work in the algebra $\tilde{\mathbf{C}}$, which is the projection of $\mathbf{C}$ onto its first l coordinates. We write the elements of $\tilde{\mathbf{C}}$ as ordered l-tuples. In this notation, the condition on $g(0)$ and $g(1)$ above tell us the elements $\vec{a} = \langle a, a, \ldots, a \rangle$ and $\vec{b} = \langle b, b, \ldots, b \rangle$ are members of $\tilde{\mathbf{C}}$, and our objective is to prove that $\vec{c} = \langle c, c, \ldots, c \rangle$ is a member of $\tilde{\mathbf{C}}$ as well. Note also that $\mathbf{3} \notin \boldsymbol{H}(\mathbf{C}) \supseteq \boldsymbol{H}(\tilde{\mathbf{C}})$.

Summarizing, we need to prove the following claim:

(†) If $\tilde{\mathbf{C}} \leqslant \mathbf{N}^l$ is subdirect, $\mathbf{3} \notin \boldsymbol{H}(\tilde{\mathbf{C}})$ and $\vec{a}$, $\vec{b} \in \tilde{C}$,

then $\vec{c} \in \tilde{C}$.

We proceed by induction on l. If $l = 1$, then $\tilde{\mathbf{C}} = \mathbf{N}$, so the element $\vec{c} = \langle c \rangle$ must exist. Suppose that $l = 2$. There are congruences α and β on $\mathbf{N}$ such that $0 \equiv a \equiv b$, $c \equiv 1 (\bmod \alpha)$, $0 \equiv c$, $a \equiv b \equiv 1 \pmod{\beta}$ and $\mathbf{N}/\alpha \cong \mathbf{N}/\beta \cong \mathbf{2}$. Since $\tilde{\mathbf{C}}$ is a subdirect square, the elements $\langle 0, 0 \rangle$ and $\langle 1, 1 \rangle$ lie in $\tilde{\mathbf{C}}$. Consider the quotient algebra $\tilde{\mathbf{C}}/(\alpha_0 \wedge \beta_1)$. (Here, $\langle x_0, x_1 \rangle \equiv \langle y_0, y_1 \rangle \pmod{\alpha_0 \wedge \beta_1} \Leftrightarrow x_0 \alpha y_0 \,\&\, x_1 \beta y_1$.) Note that $\tilde{\mathbf{C}}/(\alpha_0 \wedge \beta_1) \leqslant (\mathbf{N}/\alpha) \times (\mathbf{N}/\beta) \cong \mathbf{2}^2$. Under the canonical projection $\tilde{\mathbf{C}} \twoheadrightarrow \tilde{\mathbf{C}}/(\alpha_0 \wedge \beta_1)$, the elements $\langle 0, 0 \rangle$, $\langle a, a \rangle$, $\langle 1, 1 \rangle$ map to a 3-element chain. Since $\mathbf{3} \notin \boldsymbol{H}(\tilde{\mathbf{C}})$, $\tilde{\mathbf{C}}$ must contain an element x such that $x/(\alpha_0 \wedge \beta_1)$ is the complement of $\langle a, a \rangle/(\alpha_0 \wedge \beta_1)$. That is

$$\exists x \in \tilde{C}, \quad x \in \{\langle 1, 0 \rangle, \langle 1, c \rangle, \langle c, 0 \rangle, \langle c, c \rangle\}.$$

Similarly, by considering the congruence $\beta_0 \wedge \alpha_1$,

$$\exists y \in \tilde{C}, \quad y \in \{\langle 0, 1 \rangle, \langle c, 1 \rangle, \langle 0, c \rangle, \langle c, c \rangle\}.$$

Our goal is to show $\langle c, c \rangle \in \tilde{C}$. So suppose instead that $\langle c, c \rangle \notin \tilde{C}$. Since $\tilde{\mathbf{C}}$ is subdirect, there are $w, v \in N$ such that $\langle c, v \rangle$, $\langle w, c \rangle \in \tilde{C}$. We consider the possibilities for x and y. If $x = \langle c, 0 \rangle$, then $\langle 0, c \rangle \notin \tilde{C}$, else $x \vee \langle 0, c \rangle = \langle c, c \rangle$. Now $x \vee \langle w, c \rangle$ is equal to either $\langle c, c \rangle$ or $\langle 1, c \rangle$. Since $\langle c, c \rangle$ is excluded, we have $\langle 1, c \rangle \in \tilde{C}$. Therefore $y \neq \langle c, 1 \rangle$ (else $y \wedge \langle 1, c \rangle = \langle c, c \rangle$). Thus, $y = \langle 0, 1 \rangle$. But then $\langle 1, c \rangle \wedge y = \langle 0, c \rangle \in \tilde{C}$, which we have ruled out. Therefore, $\langle c, 0 \rangle \notin \tilde{C}$, and dually, $\langle 1, c\rangle \notin \tilde{C}$. So we are left with $x = \langle 1, 0\rangle$. But now, $x \wedge \langle c, v\rangle = \langle c, 0\rangle \in \tilde{C}$, a contradiction.

Finally, assume that $l > 2$ and that (†) holds for all smaller values of l. Let $m(x, y, z)$ denote the majority term $(x \vee y) \wedge (x \vee z) \wedge (y \vee z)$. By the induction hypothesis, the projection of $\tilde{\mathbf{C}}$ onto any set of $l-1$ coordinates contains an $(l-1)$-tuple with constant values c. Therefore, there are elements x_0, x_1, x_2 in $\tilde{C}$ such that the ith coordinate of x_j is c, for $i \neq j$. Then the element $m(x_0, x_1, x_2) = \vec{c}$ as desired. □

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