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Amalgamation classes of some distributive varieties

CLIFFORD BERGMAN

Abstract. This paper investigates the amalgamation classes of finitely generated varieties with distributive congruence lattices. Necessary and sufficient conditions are given for an algebra to be a member of the amalgamation class of a variety generated by a finite modular lattice or pseudocomplemented distributive lattice and of a filtral variety.

1. Introduction

A classical group-theoretic result due to Schreier asserts the existence of "free products with amalgamated subgroup", a construction that has found numerous uses in group theory over the years. In the early 60's, B. Jónsson directed attention to this property by showing that it could be used to construct universal and homogeneous models of certain first order theories [10, 11].

In its general form, the *amalgamation property for* $\mathcal{K}$ asserts that for any structures $A, B_0, B_1 \in \mathcal{K}$ and embeddings α_i of A into B_i $(i=0,1)$, there is a $C \in \mathcal{K}$ and embeddings β_i of B_i into C such that $\beta_1 \circ \alpha_1 = \beta_0 \circ \alpha_0$. Here $\mathcal{K}$ might be any interesting class of structures, but most often is a variety of algebras.

A good deal of research has been done on the amalgamation property (A.P.). The great majority of the results are negative, in other words, the class $\mathcal{K}$ under consideration fails to have A.P. In light of this state of affairs, it is natural to ask how close a class comes to having A.P. In [7] Grätzer and Lakser defined the amalgamation class of $\mathcal{K}$ to consist of those members of $\mathcal{K}$ that can always be "amalgamated" (see Section 2 for the definition). Since then a number of papers have appeared providing results on the amalgamation class. But even these are mostly negative. For example, Grätzer, Lakser and Jónsson [8] showed that the amalgamation class of $\mathcal{M}$ (the variety of all modular lattices) contains no non-trivial distributive lattice. In fact, there is no known example of a non-trivial lattice which is in the amalgamation class of $\mathcal{M}$. On the other hand Yasuhara [20]

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(see also Robinson [18]) proved that the amalgamation class of *every* variety is cofinal in the variety. Thus it must be quite large.

The situation was somewhat better in a number of "very small" varieties. Grätzer and Lakser [pseudocomplemented distributive lattices, 7], Quackenbush [quasiprimal algebras, 16] and Fried, Grätzer and Lakser [the lattice M_n, 4] found some sufficient conditions for an algebra to be in the amalgamation class that are necessary and sufficient when applied to a finite algebra. In this paper the study of these cases is completed and extended to a larger collection of related varieties.

Since the property of being a member of an amalgamation class is relative to the variety in question, it is not likely that a characterization could be in terms of an intrinsic property of the algebra. Theorem 3.6 (below) states that a simple algebra S is in the amalgamation class of $\mathcal{V}$ if and only if any two maximal essential extensions of S are isomorphic over S. Modulo this description, Corollary 4.9 provides an intrinsic condition for an algebra A (in a filtral variety) to be in the amalgamation class of $\mathcal{V}$.

Section 2 of this paper develops the universal algebra notions needed for the remainder. Section 3 contains some basic results on the amalgamation class. Section 4 contains the major results on filtral varieties and varieties generated by a finite modular lattice. Finally, in Section 5 we show how to modify the previous arguments to handle pseudocomplemented distributive lattices.

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2. Preliminaries

We review here the universal algebraic facts and definitions needed in the subsequent sections. Burris and Sankappanavar [2] can serve as a reference for concepts not developed here. Grätzer's volume [6] will also suffice, although its emphasis is not on varieties.

Both algebras and their underlying sets will be denoted by captial Roman letters. Lower case Greek letters denote homomorphisms between algebras. The symbolism $\alpha : A \rightarrowtail B$ indicates an embedding (one-to-one homomorphism) from A to B, and $\beta : A \twoheadrightarrow B$ indicates a surjective homomorphism.

For an algebra A, Con (A) denotes its lattice of congruences. The meet of two congruences Θ and Ψ of A is $\Theta \cap \Psi$, and the lower and upper bounds of Con (A) are Δ and ∇ (or Δ_A and ∇_A, if necessary). If $\alpha : A \to B$ and $\Theta \in \mathrm{Con}\,(B)$ then $\alpha^{-1}(\Theta) = \{(a, b) \in A^2 : \alpha(a) \equiv \alpha(b) \pmod{\Theta}\}$ is a congruence on A. In the special case that α is an embedding we write $\Theta \restriction A$ for $\alpha^{-1}(\Theta)$ if no confusion will result.

For $\Psi \in \mathrm{Con}(B)$ and $b \in B$, the coset of b modulo Ψ is written b/Ψ. The *kernel* of a homomorphism $\alpha : A \to B$ is denoted $\ker(\alpha)$ and there is an induced embedding $\alpha^* : A/\ker(\alpha) \rightarrowtail B$ given by $\alpha^*(a/\ker(\alpha)) = \alpha(a)$. In a similar fashion, if $\alpha^{-1}(\Psi) = \Theta$, there is an embedding $\alpha/\Psi : A/\Theta \rightarrowtail B/\Psi$ mapping a/Θ to $\alpha(a)/\Psi$.

A *variety* of algebras is a class closed under the formation of subalgebras, products and homomorphic images. Varieties are ordered by inclusion, and the intersection of any family of varieties is again a variety. Thus we can define *the variety generated by* $\mathcal{K}$, $V(\mathcal{K})$, as the intersection of all varieties containing $\mathcal{K}$.

We say that a variety $\mathcal{V}$ is *finitely generated* if there is a finite set $\mathcal{K}$ of finite algebras such that $\mathcal{V} = V(\mathcal{K})$.

As usual, the operators H, S, P, Pu, I denote closure under the formation of homomorphic images, subalgebras, products, ultraproducts and isomorphisms respectively. All operators are considered algebraic in the sense that $O(\mathcal{K})$ is closed under isomorphism for any class $\mathcal{K}$ and operator O. Birkhoff's well-known theorem states that $V(\mathcal{K}) = HSP(\mathcal{K})$.

Let A be an algebra and a, b distinct elements of A. Then A is (a, b)-*irreducible* if for every non-trivial congruence Θ (that is, different from Δ), $a \equiv b \pmod{\Theta}$. A is *subdirectly irreducible* if it is (a, b)-irreducible for some pair a, b, and A is *simple* if $\mathrm{Con}(A) = \{\Delta, \nabla\}$. A variety $\mathcal{V}$ is *semi-simple* if every subdirectly irreducible algebra is simple and *hereditarily simple* if every subalgebra of a subdirectly irreducible algebra is simple or trivial. The class of subdirectly irreducible members of $\mathcal{K}$ is written $\mathcal{K}_{SI}$.

Let B be a subalgebra of a product $\prod(B_j : j \in J)$, and let π_j be the coordinate projection of B to B_j. B is a *subdirect product* of the family $(B_j : j \in J)$ if π_j is surjective, for all $j \in J$. For a class $\mathcal{K}$, let $Ps(\mathcal{K})$ denote the collection of all algebras isomorphic to subdirect products of members of $\mathcal{K}$.

A *diagram in* $\mathcal{K}$ is a quintuple $\langle A, B_0, B_1, \alpha_0, \alpha_1\rangle$ with $A, B_0, B_1 \in \mathcal{K}$ and $\alpha_i : A \rightarrowtail B_i$, $i = 0, 1$. The diagram is *amalgamated in* $\mathcal{K}$ by $\langle C, \beta_0, \beta_1\rangle$ if $C \in \mathcal{K}$, $\beta_i : \beta_i \rightarrowtail C$, $i = 0, 1$ and $\beta_0 \circ \alpha_0 = \beta_1 \circ \alpha_1$. An algebra A is an *amalgamation base* of $\mathcal{K}$ if every diagram $\langle A, B_0, B_1, \alpha_0, \alpha_1\rangle$ in $\mathcal{K}$ can be amalgamated in $\mathcal{K}$. The *amalgamation class of* $\mathcal{K}$, AMAL $(\mathcal{K})$, is the collection of all amalgamation bases of $\mathcal{K}$. Finally, $\mathcal{K}$ has the *amalgamation property* (A.P.) if AMAL $(\mathcal{K}) = \mathcal{K}$.

An algebra is (*congruence*) *distributive* if its congruence lattice is distributive. A variety is distributive if every member is distributive. The results of this paper rely heavily on the following fact.

THEOREM 2.1 (Jónsson [12]). *Let $\mathcal{V}$ be a distributive variety, $A_i \in \mathcal{V}$, $i \in I$, B a subalgebra of $\prod(A_i : i \in I)$ and Θ a completely meet irreducible congruence on B. Then there is an ultrafilter D on I such that the congruence induced on B by D is contained in Θ.*

The congruence induced on B by an ultrafilter D will be denoted D_B. Some additional facts involving ultraproducts are collected below. Proofs and additional information can be found in [2, chap. 4, sec. 6].

PROPOSITION 2.2. 1. *If $\mathcal{K}$ is a finite set of finite algebras, then* $Pu(\mathcal{K}) = I(\mathcal{K})$.

2. *Let $\mathcal{V}$ be a distributive variety generated by a set $\mathcal{K}$. Then* $\mathcal{V}_{SI} \subseteq HSPu(\mathcal{K})$.

3. *If $\mathcal{V}$ is a finitely generated distributive variety then up to isomorphism, $\mathcal{V}_{SI}$ has only finitely many members and they are all finite.*

Let λ be a cardinal. A variety $\mathcal{V}$ is *residually less than* λ if every subdirectly irreducible member of $\mathcal{V}$ has cardinality less than λ. $\mathcal{V}$ *is residually small* if it is residually less than λ for some λ.

Let $A \subseteq B$. B is an *essential extension* of A if $\Theta \in \mathrm{Con}(B)$ and $\Theta \neq \Delta_B$ implies $\Theta \restriction A \neq \Delta_A$. B is a *maximal essential extension of* A *in* $\mathcal{V}$ if $B \in \mathcal{V}$ and B has no proper extension in $\mathcal{V}$ which is an essential extension of A. If $\alpha : A \rightarrowtail B$ and B is an essential extension of the image of α, then α is said to be an essential embedding. The following theorem of Taylor's is a fundamental one on residual smallness.

THEOREM 2.3 [19]. *A variety $\mathcal{V}$ is residually small if and only if every member of $\mathcal{V}$ has a maximal essential extension in $\mathcal{V}$.*

An algebra $B \in \mathcal{V}$ is an *absolute retract of* $\mathcal{V}$ if, for every extension C of B with $C \in \mathcal{V}$, there is a map ρ (a retraction) from C onto B such that ρ is the identity on B. The next proposition summarizes the relationships between these concepts.

PROPOSITION 2.4. *Let $\mathcal{V}$ be a variety, $A, B \in \mathcal{V}$ and $\alpha : A \rightarrowtail B$.*

1. *If A is (a, b)-irreducible and B is an essential extension of A, then B is $(\alpha(a), \alpha(b))$-irreducible.*

2. *For any $\Theta \in \mathrm{Con}(B)$, there is $\Psi \in \mathrm{Con}(B)$ such that $\Theta \restriction A = \Psi \restriction A$ and the induced embedding $\alpha/\Psi : A/(\Psi \restriction A) \rightarrowtail B/\Psi$ is an essential extension. In particular if $\Theta \restriction A$ is completely meet-irredicuble in $\mathrm{Con}(A)$ then Ψ is completely meet-irreducible in $\mathrm{Con}(B)$.*

3. *If B is a maximal essential extension in $\mathcal{V}$ of some algebra C, then B is an absolute retract in $\mathcal{V}$.*

4. *If B is an absolute retract in $\mathcal{V}$, then B is a maximal essential extension of itself.*

Statement 1 of this proposition says that an essential extension of a subdirectly irreducible algebra is again subdirectly irreducible. One should note however that the converse is false: an extension of one subdirectly irreducible algebra by another is not necessarily an essential extension. Thus we define a *maximal irreducible algebra* of $\mathcal{V}$ to be a subdirectly irreducible algebra of $\mathcal{V}$ with no essential extensions in $\mathcal{V}$. The class of maximal irreducibles is denoted $\mathcal{V}_{\mathrm{MI}}$.

Let B be a member of a variety $\mathcal{V}$. B is *congruence extensile in* $\mathcal{V}$ if for every extension C of B with $C \in \mathcal{V}$ and every $\Theta \in \mathrm{Con}(B)$, there is $\Psi \in \mathrm{Con}(C)$ with $\Psi \restriction B = \Theta$. $\mathcal{V}$ has the *congruence extension property* (C.E.P.) if every member of $\mathcal{V}$ is congruence extensile in $\mathcal{V}$. In the literature, an *algebra* is said to have the C.E.P. if for each subalgebra, the restriction operation maps the set of congruences of the algebra onto that of the subalgebra. We have no use for that notion here, but remark that a variety has C.E.P. just in case each member has C.E.P. A final observation:

PROPOSITION 2.5. *An algebra B is congruence extensile in $\mathcal{V}$ if and only if: for every $C \in P(\mathcal{V}_{\mathrm{SI}})$ extending B, and every completely meet-irreducible congruence Θ on B, there is a completely meet-irreducible congruence Ψ on C such that $\Psi \restriction B = \Theta$.*

Proof. Using Proposition 2.4(2), one sees the condition is necessary. So suppose the condition holds. Let D be an arbitrary extension of B in $\mathcal{V}$. Write D as a subdirect product of subdirectly irreducibles, and call their product C. Then $D \subseteq C$ and $C \in P(\mathcal{V}_{\mathrm{SI}})$. Let $\Phi \in \mathrm{Con}(B)$. There is a set $(\Theta_i : i \in I)$ of completely meet-irreducible congruences on B with $\Phi = \bigcap(\Theta_i : i \in I)$.

By assumption, there are congruences Ψ_i, $i \in I$ on C such that $\Psi_1 \restriction B = \Theta_i$ for all $i \in I$. Let $\Omega = [\bigcap(\Psi_i : i \in I] \restriction D$. Then $\Omega \restriction B = \bigcap(\Psi_i \restriction B : i \in I) = \Phi$ as desired. ∎

3. The amalgamation class

This section contains some basic facts about the amalgamation class, none of which require congruence distributivity. Except for 3.4–3.6, this material is not essentially new.

PROPOSITION 3.1 [8]. *Let $\mathcal{V}$ be an arbitrary variety, $A \in \mathrm{AMAL}(\mathcal{V})$, $B \in \mathcal{V}$ and $\alpha : B \rightarrowtail A$. Suppose that for every $C \in \mathcal{V}$ and $\beta : B \rightarrowtail C$ the diagram $\langle B, A, C, \alpha, \beta\rangle$ can be amalgamated in $\mathcal{V}$. Then $B \in \mathrm{AMAL}(\mathcal{V})$.*

Proof. Consider an arbitrary diagram $\langle B, C_0, C_1, \beta_0, \beta_1\rangle$ in $\mathcal{V}$. By assumption,

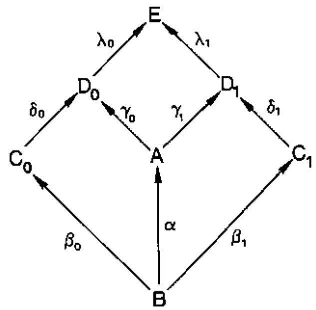

Figure 1

both of the diagrams $\langle B, A, C_0, \alpha, \beta_0\rangle$ and $\langle B, A, C_1, \alpha, \beta_1\rangle$ can be amalgamated in $\mathscr{V}$. Thus there are triples $\langle D_0, \gamma_0, \delta_0\rangle$ and $\langle D_1, \gamma_1, \delta_1\rangle$ such that $\gamma_i \circ \alpha = \delta_i \circ \beta_i$ for $i = 0, 1$. Since $A \in \mathrm{AMAL}(\mathscr{V})$, the diagram $\langle A, D_0, D_1, \gamma_0, \gamma_1\rangle$ can be amalgamated in $\mathscr{V}$, say by $\langle E, \lambda_0, \lambda_1\rangle$. A straightforward calculation will verify that $\langle E, \lambda_0 \circ \delta_0, \lambda_1 \circ \delta_1\rangle$ amalgamates the original diagram $\langle B, C_0, C_1, \beta_0, \beta_1\rangle$. (See Figure 1.) ■

PROPOSITION 3.2. *For any variety $\mathscr{V}$, every absolute retract in $\mathscr{V}$ is an amalgamation base of $\mathscr{V}$.*

Proof. Suppose A is an absolute retract in $\mathscr{V}$. Consider any diagram $\langle A, B_0, B_1, \alpha_0, \alpha_1\rangle$ in $\mathscr{V}$. Both α_0 and α_1 have left inverses, say σ_0 and σ_1. Thus $\sigma_i \circ \alpha_i = id_A$ for $i = 0, 1$. Let $C = B_0 \times B_1$, $\beta_0 : B_0 \rightarrowtail C$ by taking an element b to the pair $\langle b, \alpha_1(\sigma_0(b))\rangle$ and $\beta_1 : B_1 \rightarrowtail C$ by b goes to $\langle \alpha_0(\sigma_1(b)), b\rangle$. Then $\langle C, \beta_0, \beta_1\rangle$ amalgamates the dagram. ■

Remark. In the future it will be convenient to denote a map such as β_0 by the "product" $id_{B_0} \times (\alpha_1 \circ \sigma_0)$. Observe that in particular, Proposition 3.2 implies that every member of $\mathscr{V}_{\mathrm{MI}}$ is an amalgamation base of $\mathscr{V}$.

The following lemma contains the basic procedure for amalgamating a diagram, $\langle A, B, C, \alpha, \beta\rangle$, by writing B and C as subdirect products and working with the factors. It perhaps first appeared in [7].

LEMMA 3.3. *Let $\mathscr{V}$ be a variety and $\langle A, B_0, B_1, \alpha_0, \alpha_1\rangle$ a diagram in $\mathscr{V}$. The diagram can be amalgamated in $\mathscr{V}$ if and only if: For $\{i, j\} = \{0, 1\}$ and a, b distinct elements of B_i, there is an algebra $D \in \mathscr{V}$ and maps $\beta : B_i \to D$ and $\gamma : B_j \to D$ such that $\beta \circ \alpha_i = \gamma \circ \alpha_j$ and $\beta(a) \neq \beta(b)$.*

Proof. The condition is clearly necessary. To construct an amalgamating triple $\langle E, \delta_0, \delta_1\rangle$: take E to be the product of all of the D generated by the condition as

$\{a, b\}$ runs through all distinct pairs from B_0 and B_1, and define $\delta_i : B_i \rightarrowtail E$ to be the product of the maps β when $\{a, b\} \subseteq B_i$ and γ when $\{a, b\} \subseteq B_j$, for $i, j = 0, 1$ and $j \neq i$. ■

DEFINITION 3.4. 1. Let α_0 and α_1 be embeddings of an algebra A into B_0 and B_1 respectively. Define $\alpha_0 \cong \alpha_1$ iff there is an isomorphism γ of B_0 with B_1 such that $\gamma \circ \alpha_0 = \alpha_1$.

2. For each member A of a variety $\mathcal{V}$ let $\mathcal{E}_{\mathcal{V}}(A)$ be the class of all maximal essential embeddings of A into some algebra of $\mathcal{V}$.

In general, there is no reason to believe such maps exist, but by Theorem 2.3, if $\mathcal{V}$ is residually small, then $\mathcal{E}_{\mathcal{V}}(A)$ is always non-empty. In any case, $\cong$ is always an equivalence relation on $\mathcal{E}_{\mathcal{V}}(A)$.

3. $(A : \mathcal{V})$ is the cardinality of $\mathcal{E}_{\mathcal{V}}(A)/\cong$.

PROPOSITION 3.5. *Let A be a member of a variety $\mathcal{V}$ and* Card $(A) > 1$. *If* $A \in \text{AMAL}(\mathcal{V})$ *then* $(A : \mathcal{V}) \leq 1$.

Proof. Suppose $\alpha : A \rightarrowtail B$ and $\beta : A \rightarrowtail C$ are both members of $\mathcal{E}_{\mathcal{V}}(A)$. We need to show $\alpha \cong \beta$. By assumption, the diagram $\langle A, B, C, \alpha, \beta\rangle$ can be amalgamated by $\langle D, \alpha', \beta'\rangle$. By Proposition 2.4(3), C is an absolute retract in $\mathcal{V}$, so there is a map $\sigma : D \to C$ such that $\sigma \circ \beta' = id_C$. Let $\gamma = \sigma \circ \alpha'$. Then $\gamma \circ \alpha = \sigma \circ (\alpha' \circ \alpha) = (\sigma \circ \beta') \circ \beta = \beta$, and $\alpha^{-1}(\text{Ker } \gamma) = \text{Ker } (\gamma \circ \alpha) = \text{Ker } (\beta) = \Delta_A$. Since α is an essential embedding, we have $\text{Ker } (\gamma) = \Delta_B$ so γ is an embedding. Now since C is an essential extension of $\gamma \circ \alpha(A)$ and B is a *maximal* essential extension, γ must be an isophorphism. ■

THEOREM 3.6. *Let $\mathcal{V}$ be a residually small, semi-simple variety. Then for every $A \in \mathcal{V}_{SI} : A \in$ AMAL$(\mathcal{V})$ if and only if $(A : \mathcal{V}) = 1$.*

Proof. The implication from left to right follows from residual smallness and Proposition 3.5. Now suppose $A \in \mathcal{V}_{SI}$ and $(A : \mathcal{V}) = 1$. Let B be a maximal essential extension of A, and α the inclusion of A into B. By Proposition 2.4(3), B is an absolute retract in $\mathcal{V}$, so by 3.2, $B \in \text{AMAL}(\mathcal{V})$. We use Lemma 3.3 to show: For any $C \in \mathcal{V}$ and $\lambda : A \rightarrowtail C$, $\langle A, B, C, \alpha, \lambda\rangle$ can be amlgamated in $\mathcal{V}$. Then, by 3.1, $A \in \text{AMAL}(\mathcal{V})$.

So, write C as a subdirect product of subdirectly irreducible algebras, $C \subseteq \prod (C_j : j \in J)$. Pick $a, b \in C$ with $a \neq b$. Then there is $j \in J$ such that $\pi_j(a) \neq \pi_j(b)$. Since A is simple, the composition $\pi_j \circ \lambda$ is either one-to-one on A or collapses A to a single point. In the latter case, define $D = C_j$, $\beta = \pi_j$ and $\gamma : B \to C_j$ so that $\gamma(B) = \pi_j \circ \lambda(A)$. In the former, let D be a maximal essential extension of C_j (which exists because $\mathcal{V}$ is residually small), and $\beta = \pi_j$. D is also a maximal essential extension of A so by $(A:\mathcal{V})=1$, there is an isomorphism $\gamma: B \to D$ such that $\gamma \circ \alpha = \pi_j \circ \lambda$. In either case this satisfies 3.3. Since λ is an embedding, some $\pi_j \circ \lambda$ is one-to-one and so at least one such isomorphism γ will be produced. Since this separates any two points of B, it satisfies the condition of 3.3 for any $a, b \in B$ with $a \neq b$. ■

The above theorem gives a description of the class $\mathcal{V}_{SI} \cap \text{AMAL}(\mathcal{V})$ (when $\mathcal{V}$ is semi-simple and residually small) that forms the basis of later characterizations. Because it is referred to often, we give this class a name: $\mathcal{V}_{ASI} = \mathcal{V}_{SI} \cap \text{AMAL}(\mathcal{V})$.

COROLLARY 3.7. *If $\mathcal{V}$ is a semi-simple, residually small varity, then* $\mathcal{V}_{ASI} = \text{AMAL}(\mathcal{V}_{SI})$.

Proof. Suppose $A \in \mathcal{V}_{ASI}$. Let $\langle A, C, D, \alpha, \beta \rangle$ be a diagram in $\mathcal{V}_{SI}$. Let C' and D' be maximal essential extension of C and D. By semi-simplicity, C' and D' are both maximal essential extensions of A. From Theorem 3.6, there must be an isomorphism $\gamma: C' \to D'$ such that $\gamma \circ \alpha = \beta$. Then $\langle D', \gamma, id_D \rangle$ amalgamates the diagram in $\mathcal{V}_{SI}$.

Conversely, if $A \in \text{AMAL}(\mathcal{V}_{SI})$ then $(A:\mathcal{V})=1$, since if C and D are maximal essential extensions of A, the algebra amalgamating $\langle A, C, D, id_A, id_A \rangle$ must be isomorphic to both C and D. ■

This section concludes with three technical lemmas needed in the sequel.

LEMMA 3.8. *Let $\mathcal{V}$ be a residually small variety, $A \in \mathcal{V}$. Suppose that for every $B \in \mathcal{V}$ extending A and every $M \in \mathcal{V}_{MI}$ any homomorphism from A to M can be extended to a homomorphism from B to M. Then* $A \in \text{AMAL}(\mathcal{V})$.

Proof. Let $\langle A, B_0, B_1, \alpha_0, \alpha_1 \rangle$ be any diagram in $\mathcal{V}$. We apply Lemma 3.3. Let $a, b \in B_0$ and $a \neq b$. By writing B_0 as a subdirect product of subdirectly irreducible algebras, we can find a map $\eta: B_0 \to C$ such that $C \in \mathcal{V}_{SI}$ and $\eta(a) \neq \eta(b)$. By residual smallness, C has a maximal essential extension M which is consequently maximal irreducible. By assumption, the map $(\eta \circ \alpha_0): A \to M$ can be extended to a homomorphism γ from B_1 to M. That is, $\gamma \circ \alpha_1 = \eta \circ \alpha_0$. Taking the D of Lemma 3.3 to be M, and β to be η one easily verifies that the requirements of 3.3 are satisfied. ■

Lemma 3.8 has a converse in a special case.

PROPOSITION 3.9. *Let $\mathcal{V}$ be a residually small variety such that every* *member of $P(\mathcal{V}_{MI})$ is congruence extensile. Then $A \in \mathrm{AMAL}(\mathcal{V})$ if and only if: for every $B \in \mathcal{V}$ extending A and every $M \in \mathcal{V}_{MI}$, any homomorphism from A to M can be extended to a homomorphism from B to M.*

Proof. One direction follows from the previous lemma. So let $B \in \mathcal{V}$ extend A, M be a member of $\mathcal{V}_{MI}$ and $\alpha : A \to M$. Since $\mathcal{V}$ is residually small, A can be embedded into a product of maximal irreducible algebras. Call that product C and the embedding β. Since $A \in \mathrm{AMAL}(\mathcal{V})$, the diagram $\langle A, B, M \times C, id_A, \alpha \times \beta \rangle$ can be amalgamated by $\langle D, \delta, \gamma \rangle$.

Let Θ be the kernel of the projection of $M \times C$ onto M. By assumption, $M \times C$ is congruence extensile, so there is a congruence Ψ on D with $\gamma^{-1}(\Psi) = \Theta$. Furthermore, by 2.4, Ψ can be chosen so that the induced map $\gamma/\Psi : M \rightarrowtail D/\Psi$ is essential. Since M is *maximal* this means that γ/Ψ is invertible. Then the composition $B \to D \to D/\Psi \to M$ yields the desired extension of α. ■

LEMMA 3.10. *Let $\mathcal{V}$ be a residually small variety and $A \in \mathcal{V}$. Suppose that A is a subdirect product of algebras $(B_j : j \in J)$. Then the following three conditions are sufficient for $A \in \mathrm{AMAL}(\mathcal{V})$:*

(i) $B = \prod (B_j : j \in J) \in \mathrm{AMAL}(\mathcal{V})$

(ii) *A is congruence extensile in $\mathcal{V}$.*

(iii) *For every $M \in \mathcal{V}_{MI}$, any homomorphism from A to M extends to B.*

Proof. By 3.1 and (i), it suffices to show that for every $C \in \mathcal{V}$ and $\beta : A \rightarrowtail C$, the diagram $\langle A, B, C, id_A, \beta \rangle$ can be amalgamated in $\mathcal{V}$. Unlike the preceding lemma, this situation is not symmetrical. Using the argument there (and condition (iii)) we can find D and maps β and γ whenever $a, b \in C$ and $a \neq b$. Now suppose $a, b \in B$ and are distinct. There is a $j \in J$ such that $\pi_j^B(a) \neq \pi_j^B(b)$. Since A is a subdirect product of B, $\pi_j \restriction A$ maps A onto B_j. Let $\Theta = \mathrm{Ker}(\pi_j \restriction A)$. By (ii) Θ extends to a congruence Ψ on C, and the map β/Ψ embeds A/Θ into C/Ψ. But $A/\theta \cong B_j$, and therefore there is a map of B_j into C/Ψ. Setting D (in Lemma 3.3) equal to C/Ψ we can fulfill the requirements of 3.3. ■

4. Semi-simple varieties

In this section major results are obtained for varieties that are distributive, finitely generated and semi-simple. That the structure of these varieties is so easily understood is partially explained by the following simple lemma.

LEMMA 4.1. *Let $\mathcal{V}$ be congruence distributive, semi-simple and finitely generated. Let $(A_i : i \in I)$ be a sequence from $\mathcal{V}_{SI}$ and let Θ be a congruence on*

$A = \prod(A_i : i \in I)$. *The following are equivalent:*

(i) Θ *is a co-atom of* $\mathrm{Con}(A)$

(ii) Θ *is completely meet-irreducible in* $\mathrm{Con}(A)$

(iii) Θ *is induced by an ultrafilter over* I.

Proof. The equivalence of (i) and (ii) is merely the assertion that $\mathcal{V}$ is semi-simple (and has nothing to do with the fact that A is a product). Now assume (ii). By 2.1, there is an ultrafilter D on I such that $D_A \subseteq \Theta$, and by 2.2(3), $\mathcal{V}_{SI}$ contains only finitely many non-isomorphic members. Thus 2.2(1) applies to the ultraproduct A/D_A (with $\mathcal{K} = \mathcal{V}_{SI}$), and we conclude that A/D_A is isomorphic to some A_i, hence is simple. Finally since $\Theta \supseteq D_A$, A/Θ is a homomorphic image of A/D_A. Now, $\Theta \neq \nabla_A$ so A/Θ is nontrivial. Thus by simplicity, we conclude $\Theta = D_A$, proving (iii) holds. Conversely, each ultrafilter D on I gives rise to a simple quotient A/D_A. Thus the congruence D_A must be a co-atom of $\mathrm{Con}(A)$. Thus (iii) implies (i). ■

For elements x and y of a product such as the algebra A of Lemma 4.1, it is convenient to write $[\![x = y]\!]$ in lieu of $\{i \in I : x_i = y_i\}$.

LEMMA 4.2. *Let $\mathcal{V}$ be distributive, semi-simple and finitely generated. Let A be a product of simple algebras of $\mathcal{V}$. Then the congruences of A permute.*

Proof. Let $A = \prod(A_i : i \in I)$, each A_i a simple algebra. Let Θ and Ψ be congruences on A. Write $\Theta = \bigcap(\Theta_j : j \in J)$ and $\Psi = \bigcap(\Psi_k : k \in K)$ in which each Θ_j and Ψ_k is a completely meet-irreducible congruence of A. By 4.1, there are ultrafilters D_j and E_k such that $\Theta_j = (D_j)_A$ and $\Psi_k = (E_k)_A$ for all $j \in J$ and $k \in K$.

Now let $a, b, c \in A$ and suppose $a \equiv b \pmod{\Theta}$ and $b \equiv c \pmod{\Psi}$. Then for every $j \in J$, $a \equiv b \pmod{\Theta_j}$, so $[\![a = b]\!] \in D_j$. Similarly, for each $k \in K$, $[\![b = c]\!] \in E_k$. Now define the element d of A as follows: for $i \in [\![b = c]\!]$ set $d_i = a_i$, otherwise set $d_i = c_i$. One easily checks that $a \equiv d \pmod{\Psi}$ and $d \equiv c \pmod{\Theta}$. Since a and c were arbitrary, we have $\Theta \circ \Psi \subseteq \Psi \circ \Theta$ and by symmetry, the lemma follows. ■

PROPOSITION 4.3. *Let $\mathcal{V}$ be distributive, finitely generated and semi-simple. Suppose $A \in P(\mathcal{S})$ where $\mathcal{S} \subseteq \mathcal{V}_{SI}$. Then for every $\Theta \in \mathrm{Con}(A)$, if A/Θ is finite then $A/\Theta \in P(\mathcal{S})$.*

Proof. Write $\Theta = \bigcap(\Psi_i : i \in I)$, in which the Ψ_i are pairwise distinct, completely meet-irreducible congruences of A. Since A/Θ is finite, $\mathrm{Con}(A)$ has only finitely many members greater than Θ. Thus we may take I to be finite. By 4.2, the congruences of A permute so for each $i \in I$, $\Psi_i \circ (\bigcap(\Psi_j : j \neq i)) = \Psi_i + \bigcap(\Psi_j : j \neq i) = \bigcap(\Psi_i + \Psi_j : j \neq i) = \nabla$ using the distributivity of $\mathrm{Con}(A)$ and the fact that Ψ_i is a co-atom of $\mathrm{Con}(A)$. (We have also adapted the convention that an intersection of an empty set of congruences is equal to the largest congruence.) Therefore $A/\Theta \cong \prod (A/\Psi_i : i \in I)$. Finally, by 4.1 and 2.2(1) again, every $A/\Psi_i \in I(\mathcal{S})$. ■

The next proposition is at the heart of this investigation. As an aid to comprehension and for use in Section 5, the proof is broken into a series of numbered steps. The reader may find it helpful to draw a diagram such as Figure 2.

PROPOSITION 4.4. *Let $\mathcal{V}$ be congruence distributive, semi-simple and finitely generated. Suppose in addition that:*

Every product of maximal irreducible algebras in $\mathcal{V}$ is congruence extensile. (1)

Then every amalgamation base of $\mathcal{V}$ is a subdirect product of members of $\mathcal{V}_{\mathrm{ASI}}$.

Proof. 1. Let $A \in \mathrm{AMAL}(\mathcal{V})$. by 2.2 $\mathcal{V}_{\mathrm{SI}}$ has only finitely many members (up to isomorphism). Choose a set $\mathcal{S}$ from $\mathcal{V}_{\mathrm{SI}}$, minimal under inclusion, so that A is a subdirect product of members of $\mathcal{S}$. We will show $\mathcal{S} \subseteq \mathcal{V}_{\mathrm{ASI}}$. By 3.6 it suffices to show that for each $T \in \mathcal{S}$, $(T:\mathcal{V}) = 1$. For this, let $\alpha_0 : T \to L_0$, and $\alpha_1 : T \to L_1$ be members of $\mathcal{E}_{\mathcal{V}}(T)$. We must show $\alpha_0 \cong \alpha_1$.

2. Let $A \subseteq \prod (S_j : j \in J)$ be a subdirect product representation of A in terms of members of $\mathcal{S}$. For each $j \in J$, let π_j be the projection of A onto S_j and let $\Theta_j = \mathrm{Ker}(\pi_j)$. Define $J' = \{j \in J : S_j \neq T\}$ and $\Theta = \bigcap (\Theta_j : j \in J')$. By the minimality of $\mathcal{S}$, $\Theta \neq \Delta$. Thus there is $k \in J - J'$, such that $\Theta \nsubseteq \Theta_k$. Fix k for the remainder of the proof.

Since T is finite, there is a finite set G of elements of A, representing the cosets of Θ_k in A. Since $\Theta \nsubseteq \Theta_k$, there are elements $a, b \in A$ such that $a \equiv b(\Theta)$

$$\begin{tikzcd}[column sep=small]
 & & & E/\Psi & & & \\
 & & & E \arrow[u] & & & \\
L_0 \arrow[uurrr, "\mu_0'"] & B_0 \arrow[l, "\sigma_0"'] \arrow[urr, "\mu_0"] & & & & B_1 \arrow[ull, "\mu_1"'] \arrow[r, "\sigma_1"] & B_1/\Psi_1 \arrow[uulll, "\mu_1'"'] \\
 & & & A \arrow[ull, "\phi_0"] \arrow[urr, "\phi_1"'] \arrow[d, "\pi_k"] & & & \\
 & & & T \arrow[uulll, "\alpha_0"] \arrow[uurrr, "\phi_1'"'] & & &
\end{tikzcd}$$

Figure 2a

$$\begin{tikzcd}
B_1 \arrow[r, "\sigma_1"] & B_1/\Psi_1 \arrow[r, "\gamma"] & L_1 \\
A \arrow[u, "\phi_1"] & & \\
F \arrow[u] \arrow[r, "\pi_k"'] & T \arrow[uur, "\alpha_1"'] &
\end{tikzcd}$$

Figure 2b

but $a \not\equiv b(\Theta_k)$. Let F be the subalgebra of A generated by $G \cup \{a, b\}$. Since $\mathcal{V}$ is finitely generated, F must be finite. The map $\pi_k \restriction F$ maps F onto T since F contains the members of G which map to T. Let $\Phi = \Theta_k \restriction F$. Since $a, b \in F$, $\Theta \restriction F \nsubseteq \Phi$.

3. Define $R = \{j \in J : \Theta_j \restriction F = \Phi\}$ and $Q = J - R$. Then $J' \subseteq Q$. (Proof: if $j \in J'$ then $\Theta \subseteq \Theta_j$ so $\Theta \restriction F \subseteq \Theta_j \restriction F$. Since $\Theta \restriction F \nsubseteq \Phi$, we cannot have $\Theta_j \restriction F = \Phi$.) For each $j \in R$, we have an isomorphism $\delta_j : S_j \to S_k$ given by

$$\delta_j = \pi_k \restriction F \circ (\pi_j \restriction F)^{-1}. \tag{2}$$

This map is well defined since $\mathrm{Ker}\,(\pi_j \restriction F) = \mathrm{Ker}\,(\pi_k \restriction F) = \Phi$.

Now, for each $j \in Q$, fix a maximal essential extension N_j of S_j. We define structures B_0 and B_1 by: $B_i = \prod (L_i : j \in R) \times \prod (N_j : j \in Q)$ for $i = 0, 1$. There are embeddings ϕ_0, ϕ_1 of A into B_0 and B_1 given coordinatewise by $\phi_0 = (\prod (\alpha_0 : j \in R) \times \prod (id_{S_j} : j \in Q)) \restriction A$, $\phi_1 = (\prod (\alpha_1 \circ \delta_j : j \in R) \times \prod (id_{S_j} : j \in Q)) \restriction A$. Since $A \in \mathrm{AMAL}\,(\mathcal{V})$, the diagram $\langle A, B_0, B_1, \phi_0, \phi_1 \rangle$ can be amalgamated in $\mathcal{V}$ by $\langle E, \mu_0, \mu_1 \rangle$. Let σ_0 be the kth coordinate projection of B_0 onto L_0, and let $\Psi_0 = \mathrm{Ker}\,(\sigma_0)$. Observe that $\sigma_0 \circ \phi_0 = \alpha_0 \circ \pi_k$, so that the part of Figure 2a so far constructed commutes.

5. B_0 is constructed so as to be a product of maximal irreducible algebras. By assumption (1) B_0 is congruence extensile. Thus there is a congruence Ψ on E such that $\mu_0^{-1}(\Psi) = \Psi_0$ and by 2.4(2) it can be chosen so that the induced map, $\mu_0' : L_0 \rightarrowtail E/\Psi$, is an essential extension. Since L_0 was chosen to be a maximal essential extension of T, μ_0' must be an isomorphism.

6. Let $\Psi_1 = \mu_1^{-1}(\Psi)$. Then $\mu_1' : B_1/\Psi_1 \rightarrowtail E/\Psi \cong L_0$, and since L_0 is finite, B_1/Ψ_1 must be finite. But B_1 is a product of maximal irreducible algebras, so by 4.3, B_1/Ψ_1 is again a product of maximal irreducible algebras. Applying assumption (1) again, every congruence on B_1/Ψ_1 must extend to E/Ψ. But E/Ψ is simple, so we conclude that B_1/Ψ_1 is also simple – in other words, it is a product of just one factor from B_1. In particular it is a maximal irreducible algebra of $\mathcal{V}$, so μ_1' is an isomorphism. Let $\sigma_1 : B_1 \twoheadrightarrow B_1/\Psi_1$. Observe that $\phi_1^{-1}(\Psi_1) = \phi_0^{-1}(\Psi_0) = \Theta_k$. Thus there is an induced embedding $\phi_1' : T \rightarrowtail B_1/\Psi_1$ and the entire arrangement of Figure 2a commutes. In particular $\mu_0' \circ \alpha_0 = \mu_1' \circ \phi_1'$ and since μ_0' and μ_1' are isomorphisms we conclude that $\alpha_0 \cong \phi_1'$.

7. For each $j \in J$, let Ω_j be the corresponding projection congruence on B_1. Since Ψ_1 is completely meet-irreducible in $\mathrm{Con}\,(B_1)$, Lemma 4.1 tells us that it is induced by an ultrafilter D on J. We make the following

CLAIM. $R \in D$.

Proof. For each pair $x, y \in F$ define $U_{x,y} = \{j \in J : \phi_1(x) \equiv \phi_1(y) \pmod{\Omega_j}\}$. Let $U = \bigcap \{U_{x,y} : (x, y) \in \Phi\}$. Observe that $\Psi_1 \restriction F = \Psi_0 \restriction F = \Theta_k \restriction F = \Phi$. If $x \equiv y(\Phi)$ then $U_{x,y} \in D$ since we will have $\phi_1(x) \equiv \phi_1(y) \pmod{\Psi_1}$ so $\{j \in J : \phi_1(x) \equiv \phi_1(y) \pmod{\Omega_j}\} \in D$. Since Φ is a finite set of pairs and D is closed under finite intersection, $U \in D$.

On the other hand, $j \in U$ iff $\Phi \subseteq \Omega_j \restriction F$. Since Φ is a completely meet-irreducible congruence of F, it has a unique upper cover Φ' in $\operatorname{Con}(F)$. Thus $U = \{j \in J : \Phi = \Omega_j \restriction F\} \cup \{j \in J : \Phi' \subseteq \Omega_j \restriction F\}$. The first of these sets is equal to R since $\Omega_j \restriction F = \Theta_j \restriction F$. Choose a pair $(x, y) \in \Phi' - \Phi$. Since $x \not\equiv y \pmod{\Phi}$, the set $\{j \in J : \phi_1(x) \not\equiv \phi_1(y) \pmod{\Omega_j}\}$ is a member of D and is disjoint from the latter summand of U. Thus $R \supseteq \{j \in J : \phi_1(x) \not\equiv \phi_1(y) \pmod{\Omega_j}\} \cap U \in D$, proving the claim.

8. From step 6 we have $\alpha_0 \cong \phi_1'$. Thus we can complete the proof by showing $\phi_1' \cong \alpha_1$. Now $\Psi_1 = D_{B_1}$, $R \in D$ and, for every $r \in R$, $B_1/\Omega_r \cong L_1$. Thus, using the finiteness of L_1, there is an isomorphism $\gamma : B_1/\Psi_1 \to L_1$ such that

$$\gamma(b/\Psi_1) = x \text{ if and only if } \{r \in R : \pi_r(b) = x\} \in D.$$

Then $\gamma \circ \phi_1' \circ (\pi_k \restriction F) = \gamma \circ \sigma_1 \circ \phi_1 \restriction F = \alpha_1 \circ (\pi_k \restriction F)$. Proof: only the second equality is difficult. Let $f \in F$. For every $r \in R$, $\pi_r \circ \phi_1(f) = \alpha_1 \circ \delta_r \circ \pi_r(f) = \alpha_1 \circ \pi_k(f)$ by equation (2). Let $x = \alpha_1 \circ \pi_k(f)$. Then $\{r \in R : \pi_r \circ \phi_1(f) = x\} = R \in D$, so by the definition of γ, the equality holds. See Figure 2b. Since $\pi_k \restriction F$ is an epimorphism, it cancels in the above equation yielding $\gamma \circ \phi_1' = \alpha_1$, and therefore $\phi_1' \cong \alpha_1$ as desired. ■

DEFINITION 4.5. For a variety $\mathcal{V}$ and a member A:
1. $A^{\$} = \prod (A/\Theta : \Theta \in \operatorname{Con}(A) \text{ and } A/\Theta \in \mathcal{V}_{\mathrm{ASI}})$
2. μ_A is the canonical map from A to $A^{\$}$
3. A has *property* P if: for every $M \in \mathcal{V}_{\mathrm{MI}}$ and $\alpha : A \to M$ there is a map $\beta : A^{\$} \to M$ such that $\beta \circ \mu_A = \alpha$.

Remarks. It is regrettable that neither the notation nor phase "property P" make reference to the variety $\mathcal{V}$ which plays a significant role in their interpretation. In the contexts in which these notions are likely to be used, there does not seem to be any danger of ambiguity (and results in a less burdensome notation). Observe also that (3) is generally vacuous unless $\mathcal{V}$ is residually small. When $\mathcal{V}$ is residually small, property P implies that the map μ is injective.

THEOREM 4.6. *Let $\mathcal{V}$ be distributive, semi-simple and finitely generated. Suppose further that:*

members of $P(\mathcal{V}_{\mathrm{ASI}})$ are congruence extensile. (3)

Then for $A \in \mathcal{V} : A \in \mathrm{AMAL}(\mathcal{V})$ if and only if A is congruence extensile and has property P.

Proof. Suppose first that A satisfies the right hand side. To show A is an amalgamation base, we use Lemma 3.10 with B taken to be $A^\$$. Condition (ii) holds by assumption and (iii) follows directly from property P, as does the fact that A is a subdirect product of $A^\$$ (i.e. μ is one-to-one). All that remains is to verify that $A^\$ \in \mathrm{AMAL}(\mathcal{V})$, and this is done by applying Lemma 3.8.

So let C be an arbitrary extension of $A^\$$, M a member of $\mathcal{V}_{\mathrm{MI}}$ and α a homomorphism from $A^\$$ to M. We show that α extends to a map from C to M. Let $\Theta = \mathrm{Ker}(\alpha)$. Since M is finite (Proposition 2.2) and $A^\$/\Theta$ can be embedded into M, Proposition 4.3 implies that $A^\$/\Theta \in P(\mathcal{V}_{\mathrm{ASI}})$. But by assumption (3) and the simplicity of M, we conclude that $A^\$/\Theta$ is simple, in other words $A^\$/\Theta \in \mathcal{V}_{\mathrm{ASI}}$.

On the other hand, applying (3) directly to $A^\$$, there is a congruence Ψ on C such that $\Psi \upharpoonright A^\$ = \Theta$, and by 2.4, we can take Ψ to be completely meet-irreducible. Thus $A^\$/\Theta$ can be embedded into the simple algebra C/Ψ, which in turn can be embedded into a maximal irreducible algebra L. Finally applying 3.6 to $A^\$/\Theta$ we conclude that $L \cong M$, and the desired extension of α is given by the composition $C \twoheadrightarrow C/\Psi \rightarrowtail L \cong M$. Therefore $A^\$$, hence A is an amalgamation base of $\mathcal{V}$.

For the converse, let $A \in \mathrm{AMAL}(\mathcal{V})$. By 4.4 the map μ_A is an injection. To show A is congruence extensile, let C be an arbitrary extension of A. By the remarks in Section 2, it suffices to consider a completely meet-irreducible congruence Θ on A. A/Θ is subdirectly irreducible, so it has a maximal essential extension E, which is in $\mathcal{V}_{\mathrm{ASI}}$ by 2.4 and 3.2. There is an embedding γ of A into $E \times A^\$$ given by $\gamma(a) = \langle a/\Theta, \mu(a)\rangle$.

Let Ψ be the kernel of the projection of $E \times A^\$ \twoheadrightarrow E$. Then $\gamma^{-1}(\Psi) = \Theta$. By assumption, the diagram $\langle A, E \times A^\$, C, \gamma, id_A\rangle$ can be amalgamated, say by $\langle D, \beta, \delta\rangle$. $E \times A^\$$ is congruence extensile (assumption (3) again) so there is $\Omega \in \mathrm{Con}(D)$ with $\delta^{-1}(\Omega) = \Psi$. Then the congruence $\beta^{-1}(\Omega)$ on C is an extension of Θ.

Finally, we check property P. Let M be a maximal irreducible algebra, $\alpha : A \to M$ and $\Theta = \ker(\alpha)$. Using the same construction as above with E taken to be M and C to be $A^\$$, we amalgamate the diagram $\langle A, A^\$, A^\$ \times M, \mu, \mu \times \alpha\rangle$ by $\langle D, \beta, \delta\rangle$. The congruence Ψ (still the projection of $A^\$ \times M$ onto M) extends to Ω on D. Then there is an induced embedding δ/Ω of M into D/Ω. Since M is an absolute retract, there is a (one sided) inverse mapping D/Ω back to M. We can now satisfy property P with the composition $A^\$ \rightarrowtail D \twoheadrightarrow D/\Omega \to M$. ■

Theorem 4.6 provides the basic characterization of the amalgamation class. In attempting to find specific varieties to which it applies, one must deal with the rather unfamiliar condition expressed in (3). The easiest way to handle it is to assume $\mathcal{V}$ has the congruence extension property. This leads to the following.

DEFINITION 4.7. A variety $\mathcal{V}$ is *filtral* if, for each subdirect procuct B of a family $(B_i : i \in I)$ of subdirectly irreducible algebras, and each $\Theta \in \mathrm{Con}(B)$, there is a filter D on I such that $D_B = \Theta$.

In the past few years, a great deal has been learned about filtral varieties, see Bergman [1], Köhler and Pigozzi [13], Fried and Kiss [5] and Davey [3] among others. It turns out that when C.E.P. is added to the assumption of 4.6, the resulting varieties are all filtral.

THEOREM 4.8. *Let $\mathcal{V}$ be a finitely generated variety. Then $\mathcal{V}$ is filtral if and only if it is distributive, semi-simple and has the C.E.P.*

All three conditions follow from filtrality without the assumption that $\mathcal{V}$ is finitely generated. For the proofs, see the papers referenced above.

COROLLARY 4.9. *Let $\mathcal{V}$ be a finitely generated filtral variety. Then for all $A \in \mathcal{V} : A \in \mathrm{AMAL}(\mathcal{V})$ if and only if A has property P.*

This follows immediately from 4.6 and 4.8. As an application we obtain a special case of a theorem of Grätzer–Lakser [7, Theorem 3].

COROLLARY 4.10. *Let $\mathcal{V}$ be a finitely generated filtral variety. Then $\mathcal{V}$ has the amalgamation property if and only if $\mathcal{V}_{\mathrm{SI}} \subseteq \mathrm{AMAL}(\mathcal{V})$.*

Proof. Let $A \in \mathcal{V}$. To show A has property P, let $M \in \mathcal{V}_{\mathrm{MI}}$ and $\alpha : A \to M$. Let $\Theta = \ker(\alpha)$. Since $\mathcal{V}$ has C.E.P., there is a congruence Ψ on $A^{\$}$ such that $\Psi \restriction A = \Theta$. Since α^* embeds A/Θ into M, and M is simple, $A/\Theta \in \mathcal{V}_{\mathrm{SI}} \subseteq \mathrm{AMAL}(\mathcal{V})$. Thus the diagram $\langle A/\Theta, A^{\$}/\Psi, M, \mu/\Psi, \alpha^* \rangle$ can be amalgamated in $\mathcal{V}$. As in the proof of Theorem 4.6, M is an absolute retract, so an extension of α to $A^{\$}$ can be constructed as before. ■

One final corollary regarding filtral varieties.

COROLLARY 4.11. *Let $\mathcal{V}$ be a finitely generated filtral variety. For a finite algebra $A \in \mathcal{V}$, $A \in \mathrm{AMAL}(\mathcal{V})$ if and only if $A \in Ps(\mathcal{V}_{\mathrm{ASI}})$.*

Proof. Necessity follows from 4.4 For the converse, $A \in Ps(\mathcal{V}_{ASI})$ implies that μ_A is one-to-one. To check property P, let $\alpha : A \to M$ in which $M \in \mathcal{V}_{MI}$. As $\alpha(A)$ is simple, $\ker(\alpha)$ is a coatom of $\mathrm{Con}(A)$. Using distributivity, the finiteness of $\mathrm{Con}(A)$ and injectivity of μ, $\ker(\alpha) \in \{\Theta \in \mathrm{Con}(A) : A/\Theta \in \mathcal{V}_{ASI}\}$. Therefore the composition $A^{\$} \twoheadrightarrow A/\ker(\alpha) \rightarrowtail M$ extends α. ■

Returning now to the general situation of Theorem 4.6, are there varieties lacking the C.E.P. for which assumption (3) is valid? One large class of such varieties are those generated by a finite modular lattice. Except for the variety of distributive lattices, these varieties all fail to satisfy C.E.P. since they contain the lattice M_3. They are all semi-simple (see [9]) and of course they are all congruence distributive. We verify condition (3).

LEMMA 4.12. *Let $\mathcal{V}$ be a finitely generated, semi-simple, congruence distributive variety. Suppose no non-simple member of $P(\mathcal{V}_{ASI})$ can be embedded in a simple member of $\mathcal{V}$. Then the members of $P(\mathcal{V}_{ASI})$ are all congruence extensile, that is, condition* (3) *of Theorem* 4.6 *holds.*

Proof. Let $A \in P(\mathcal{V}_{ASI})$, $B = \prod(B_i : i \in I)$ in which every B_i is simple, and let Θ be a completely meet-irreducible congruence of A. By Proposition 2.5 is suffices to show that if $A \subseteq B$ then Θ extends to B. By 2.1, there is an ultrafilter D on I such that $D_A \subseteq \Theta$. By 2.2, $B/D_B \cong B_i$ some $i \in I$. Thus there is an induced embedding of A/D_A into B_i. Since B_i is finite, 4.3 implies that $A/D_A \in P(\mathcal{V}_{ASI})$. But by hypothesis, A/D_A must then be simple. Since $D_A \subseteq \Theta \neq \nabla$, we conclude that $D_A = \Theta$, and of course D_A extends to $D_B \in \mathrm{Con}(B)$. ■

THEOREM 4.13. *Let $\mathcal{V}$ be a finitely generated variety of modular lattices. Then for $A \in \mathcal{V} : A \in \mathrm{AMAL}(\mathcal{V})$ if and only if A is congruence extensile and has property P.*

Proof. In order to apply Theorem 4.6 we need only verify that the assumptions of Lemma 4.12 hold. Let $G \in \mathcal{V}_{SI}$ and $A = \prod(A_i : i \in I)$ in which each A_i is a member of $\mathcal{V}_{ASI}$ and $A \subseteq G$. Since G is finite, we may assume that I is finite and each A_i is a nontrivial lattice. In fact, let us assume that $I = \{0, 1, \ldots, n\}$. It will suffice for us to prove that $n = 0$.

Suppose $n > 0$. Then there are elements $a, b \in A_1$ with $a < b$. Fix elements $c_j \in A_j$ for $j = 2, 3, \ldots, n$. Define embeddings α and β from A_0 to A by $\alpha(x) = \langle x, a, c_2, c_3, \ldots, c_n\rangle$ and $\beta(x) = \langle x, b, c_2, \ldots, c_n\rangle$. By embedding G into a maximal irreducible algebra M, we see that α and β are both members of $E_{\mathcal{V}}(A_0)$. By Theorem 3.6, there is an automorphism γ of M such that $\gamma \circ \alpha = \beta$. However, for each $x \in A_0$, $\alpha(x) < \beta(x)$ in M. Since M is finite it has no automorphism taking an element to a strictly larger one. This is a contradiction, proving that $n = 0$. ∎

The following example shows that condition (3) of Theorem 4.6 can fail, even for a variety which is generated by an expansion of a modular lattice. Let M_3 be the lattice of Figure 3. Expand the language of lattice theory to include four new constant symbols: k_0, k_1, k_2, k_3. Let $M = \langle M_3, \vee, \wedge, z, a, c, u\rangle$ be an algebra in this new language, and let $\mathcal{V} = V(M)$. M has a single proper subalgebra N with universe $\{z, a, c, u\}$ and N has two proper homomorphic images: $L_1 = \langle\{0, 1\}, \vee, \wedge, 0, 0, 1, 1\}$ and $L_2 = \langle\{0, 1\}, \vee, \wedge, 0, 1, 0, 1\rangle$ in which $\langle\{0, 1\}, \vee, \wedge\}$ is the two element chain. Observe that L_1 is not isomorphic to L_2 and $N \cong L_1 \times L_2$. Thus the subdirectly irreducible algebras of $\mathcal{V}$ are M, L_1, L_2 and they are all maximal, so $\mathcal{V}_{SI} = \mathcal{V}_{MI} = \mathcal{V}_{ASI}$. On the other hand, N is not congruence extensile, so (3) fails for this variety.

Corollary 4.10 showed that for a filtral variety, $\mathcal{V}_{SI} \subseteq \mathrm{AMAL}(\mathcal{V})$ implies that $\mathcal{V}$ has A.P. The following example, suggested by R. McKenzie, shows that the conditions of Theorem 4.6 are not sufficient for this implication to hold.

Consider again the lattice M_3 of Figure 3. Expand the language to include three new constant symbols, k_0, k_1, k_2, and a new unary operation: $'$. Define this operation on M_3 by: $z' = z$, $b' = b$, $u' = u$, $a' = c$ and $c' = a$. Let $M = \langle M_3, \vee, \wedge, ', z, b, u\rangle$ and $\mathcal{V} = V(M)$. Then $\mathcal{V}$ is distributive and finitely generated. The subdirectly irreducibles (being members of $HS(M)$) are Q_0, Q_1 and M where $Q_0 = \langle\{0, 1\}, \vee, \wedge, ', 0, 0, 1\rangle$ and $Q_1 = \langle\{0, 1\}, \vee, \wedge, ', 0, 1, 1\rangle$ and in both cases, $'$ is the identity operation. Thus $\mathcal{V}$ is semi-simple.

Observe that Q_0, Q_1, and M are all maximal irreducible and therefore $\mathcal{V}_{SI} \subseteq \mathrm{AMAL}(\mathcal{V})$. On the other hand, Q_0 and Q_1 have no proper subalgebras and M has only $N = \{z, b, u\}$. Thus Lemma 4.12 applies, so $\mathcal{V}$ meets all the conditions of Theorem 4.6. But N is not simple, hence it is not congruence extensile so by 4.6, $N \notin \mathrm{AMAL}(\mathcal{V})$. Thus $\mathcal{V}$ does not have A.P.

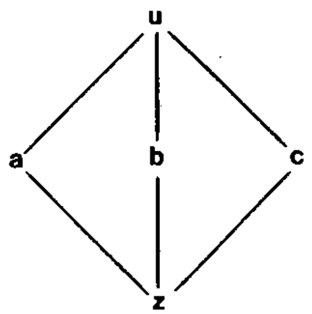

Figure 3.

As an analogue to Corollary 4.11, we have the following

PROPOSITION 4.14. *Let $\mathcal{V}$ be a finitely generated, semi-simple, distributive variety such that members of $P(\mathcal{V}_{ASI})$ are congruence extensile. Let $A \in \mathcal{V}$ be finite. The following are equivalent:*

(i) $A \in Ps(\mathcal{V}_{ASI})$ *and is congruence extensile*

(ii) A *satisfies property* P

(iii) $A \in \mathrm{AMAL}(\mathcal{V})$.

Proof. By 4.6, (iii) is equivalent to (i) and (ii) combined. Thus it suffices to show the equivalence of (i) and (ii).

Assume (i). Then in particular, the map μ_A is an embedding. Since A is finite, we can write $A^{\$} = \prod (B_i : i < n)$ in which each $B_i \in \mathcal{V}_{ASI}$. Let Θ_i be the kernel of the projection of A onto B_i for all $i < n$. Let $\gamma : A \to M$ where $M \in \mathcal{V}_{MI}$ and $\Theta = \mathrm{Ker}(\gamma)$. Using the finiteness (and distributivity) of $\mathrm{Con}(A)$, we can write $\Theta = \Theta_0 \cap \Theta_1 \cap \cdots \cap \Theta_t$ (renumbering if necessary) for some $t < n-1$. Then there is an embedding δ of A into $M \times \Pi(B_i : t < i < n)$ given by $\delta(a) = \langle \gamma(a), a/\Theta_{t+1}, a/\Theta_{t+2}, \ldots, a/\Theta_{n-1}\rangle$. Let $\Psi_0, \Psi_{t+1}, \ldots, \Psi_{n-1}$ be the projection kernels of this new product. Since A is congruence extensile, there is a congruence Φ on this algebra extending Θ_t. Again using finiteness, $\Phi = \bigcap (\Psi_j : j \in J)$ for some set $J \subseteq \{0, t+1, \ldots, n-1\}$. Thus $\Theta_t = \Phi \restriction A = \bigcap (\Psi_j \restriction A : j \in J)$. Since Θ_t is meet-irreducible, there is $j \in J$ with $\Theta_t = \Psi_j \restriction A$. If $j \neq 0$, then $\Psi_j \restriction A = \Theta_j$, and we have $t = j \in J$ which is impossible. Thus $\Theta_t = \Psi_0 \restriction A = \Theta$, and we conclude $t = 0$. In other words, $\Theta = \Theta_0$ is meet-irreducible. Therefore the induced map of A/Θ to B_0 is an isomorphism. Composing $A^{\$} \twoheadrightarrow B_0 \cong A/\Theta \rightarrowtail M$ extends γ, so (ii) is proved.

For the converse, it has already been noted that property P implies that μ_A is injective, hence $A \in P(\mathcal{V}_{ASI})$. Observe further that the finiteness of A insures that every one of its completely meet-irreducible congruences is a member of $\{\Theta \in \mathrm{Con}(A) : A/\Theta \in \mathcal{V}_{ASI}\}$, so they all extend to (coordinate projection) congruences on $A^{\$}$.

In order to show that A is congruence extensile it suffices to consider au extension C of A in which C is a product of maximal irreducible algebras $(C_j : j \in J)$. For each $j \in J$ there is a map $\gamma_j : A \to C_j$. By property, P, γ_j extends to $\delta_j : A^{\$} \to C_j$. Let $\Psi_j = \ker(\delta_j)$ and $\Psi = \bigcap (\Psi_j : j \in J)$. Then $\Psi \restriction A = \Delta_A$ since A is embedded in C, and by the comments above (and distributivity) this means $\Psi = \Delta_{A^{\$}}$. Hence the family $(\delta_j : j \in J)$ embeds $A^{\$}$ into C. Now let Θ be completely meet-irreducible in $\mathrm{Con}(A)$. Θ extends to a congruence on $A^{\$}$, and by assumption, $A^{\$}$ is congruence extensile. Thus Θ can be extended all the way to C. ■

This proposition raises two questions. First, for a finite algebra A, does

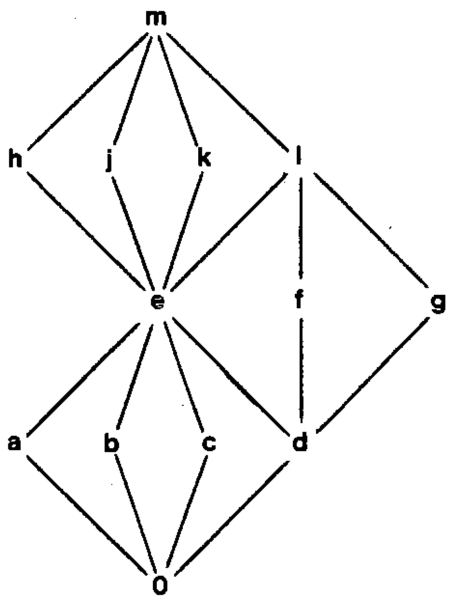

Figure 4.

$A \in Ps(\mathcal{V}_{ASI})$ imply A has property P? Secondly, can the assumption that A is finite be dropped? We sketch two examples below to show the answer in both cases is no. However, with regard to the second question, the example shows only that (i) $\not\Rightarrow$ (ii). The opposite implication is an open question.

Consider the lower bounded lattice $\langle L, \vee, \wedge, 0\rangle$ of Figure 4. Let $\mathcal{V}$ be the variety generated by L, and let M be the subalgebra with universe $\{0, a, b, c, d, e\}$. By checking the members of $HS(L)$, one can verify that the only maximal essential extension of M is L, thus $M \in \mathcal{V}_{ASI}$. Also, the two element chain is not a member of $\mathcal{V}_{ASI}$.

Therefore, cardinality requirements dictate that Lemma 4.12 applies (hence 4.6 and 4.14 as well). L has another subalgebra N with universe $L-\{f, g\}$. N is a subdirect product of two copies of M. But N is not congruence extensile, so it is not an amalgamation base.

For our second example, let $M = M_3$ once again be the lattice of Figure 3 and let $\mathcal{V} = V(M)$. Let I be an infinite set, D a nonprincipal ultrafilter over I and $\rho: M^I \twoheadrightarrow M$ be induced by D. Let $A = \rho^{-1}\{z, u\}$, i.e. those elements of M^I that are "almost always" equal to z or to u. A is a subalgebra of M^I. Observe that if E is any ultrafilter distinct from D then $A/E_A \cong M$, while $A/D_A \cong C_2$ (the two element chain). Therefore, A is a subdirect product of copies of M, in fact $A^\$ \cong M^{I^*}$, where I^* is the set of all ultrafilters on I except for D.

Furthermore, A is congruence extensile in $\mathcal{V}$. It is enough to check an extension C of the form M^J, since M is the only maximal irreducible algebra. Let Θ be a completely meet-irreducible congruence on A. By 2.1, there is an ultrafilter F on J such that $F_A \subseteq \Theta$ and there is an embedding of A/F_A into $C/F_C \cong M$. If $A/\Theta \cong M$ then we must have $A/F_A \cong M$ so $F_A \cong \Theta$. If $A/\Theta \cong C_2$ then $A/F_A \cong C_2$, C_3 or $C_2 \times C_2$. However the latter two alternatives are impossible because A has only one congruence whose quotient yields C_2. Therefore $F_A = \Theta$ again. So Θ extends to C.

On the other hand, A does not have property P. For let $\alpha: A \to M$ be such that $\alpha(x) = z$ if $\rho(x) = z$ and $\alpha(x) = a$ if $\rho(x) = u$. One easily checks that if α could be extended to $A^\$$ then the kernel of that extension would give rise to a congruence on A, distinct from D_A, such that the quotient is C_2.

As was pointed out in the introduction, the property of being an amalgamation base is not an intrinsic property of an algebra, since it depends on the variety under consideration. The characterization given in Corollary 4.9 (for filtral varieties) seems to be as close as one could hope to get. It presupposes a knowledge of the class $\mathcal{V}_{ASI}$ which, since all simple algebras are finite, should be relatively easy to compute using Theorem 3.6. One must check each simple algebra to see if it can be embedded into two non-isomorphic maximal irreducibles, or into one maximal irreducible algebra in two ways that are not connected by an automorphism. Given that, property P is essentially a condition on the congruence lattice of the algebra in question. For finite algebras, this yields an effective procedure for determining membership in AMAL $(\mathcal{V})$, especially in light of Corollary 4.11. For infinite algebras, this is not so, but as in the example discussed above, such an analysis is sometimes possible.

In the special case of discriminator varieties however, it is possible to do better. In [21] it is proved that the amalgamation class of a finitely generated discriminator variety of finite type is finitely axiomatizable.

A somewhat different way of looking at the content of 4.9, is to compare it to the more general characterization of Proposition 3.9. The improvement is that the universal quantifier of 3.9 ("for every $B \in \mathcal{V}$ extending A") is replaced by a specific algebra ($A^\$$).

For varieties of modular lattices, the situation is not so clear. Not only must one check property P, but also that the algebra is question is congruence extensile. The purpose of this final proposition is to show that this second property is also a condition on the congruence lattice.

For a variety $\mathcal{V}$ and algebra A, call a set $(\Psi_j : j \in J) \subseteq \mathrm{Con}(A)$ *a separating family of congruences* if $\bigcap (\Psi_j : j \in J) = \Delta$ and, for every $j \in J$, $A/\Psi_j \in S(\mathcal{V}_{SI})$.

PROPOSITION 4.15. *Let $\mathcal{V}$ be a finitely generated, semi-simple, distributive* *variety. An algebra $A \in \mathcal{V}$ is congruence extensile if and only if: For every coatom Θ of* Con (A) *and every separating family* $(\Psi_j : j \in J)$, *there are* $a, b \in A$ *such that for all finite sets* $F \subseteq \Theta$, *there is* $j \in J$ *with* $F \subseteq \Psi_j$ *and* $a \not\equiv b$ (mod Ψ_j).

Proof. By 2.5, it suffices to consider an extension C of A such that $C = \prod (C_j : j \in J)$, every $C_j \in \mathcal{V}_{\mathrm{SI}}$ and a coatom Θ of Con (A). Let Ψ_j be the kernel of the induced map of A to C_j. Then the family $(\Psi_j : j \in J)$ is a separating family.

For a pair of elements x and y, of A, let $C(x, y) = \{j \in J : x \equiv y \pmod{\Psi_j}\}$. Observe that the condition in the statement of the theorem on the pair (a, b) and finite set F is equivalent to: the family $\{C(x, y) : (x, y) \in \Theta\} \cup \{J - C(a, b)\}$ has the finite intersection property. This in turn is equivalent to the existence of an ultrafilter D on J, containing every one of these sets.

Suppose such an ultrafilter exists. To see $D_A = \Theta$, observe that $a \not\equiv b \pmod{D_A}$ while $x \equiv y \pmod{D_A}$ whenever $x \equiv y \pmod{\Theta}$. Thus $\Theta \subseteq D_A \neq \nabla_A$. Since Θ is a coatom, $\Theta = D_A$. Conversely, if Θ extends to a congruence Ω on C, then by 2.4 and 4.1, Ω is induced by an ultrafilter over J.

Choose any pair a, b incongruent modulo Θ. Then one easily checks that this ultrafilter contains all the necessary sets. ■

5. Pseudocomplemented distributive lattices

In [14] and [7], G. Grätzer and H. Lakser analyze the structure of varieties of pseudocomplemented distributive lattices. Among other things, they consider the amalgamation property and amalgamation class of these varieties. They show that the variety of all pseudocomplemented distributive lattices has A.P., and go a long way toward describing the amalgamation bases of its subvarieties. In this section it is shown how to modify the techniques developed above to complete the description. Our goal is to prove that for a finitely generated variety $\mathcal{V}$ of pseudocomplemented distributive lattices, $A \in \mathrm{AMAL}(\mathcal{V})$ if and only if A has property P.

DEFINITION 5.1. A *pseudocomplemented distributive lattice* is an algebra $\langle L, \wedge, +, {}^*, 0, 1\rangle$ such that $\langle L, \wedge, +, 0, 1\rangle$ is a bounded distributive lattice and * is a unary operation satisfying: $x \wedge y = 0$ if and only if $y \leq x^*$.

The class of pseudocomplemented distributive lattices is a variety, see P. Ribenboim [17]. Let B be a boolean algebra with least element 0 and largest element denoted e. Adjoin a new largest element, 1, to B, and call the result B^+. For each $x \in B^+$, define x^* to be the B-complement of x, if $x \neq 0, 1$; and define

$0^* = 1$, $1^* = 0$. Then the resulting structure $\langle B^+, \wedge, +, {}^*, 0, 1\rangle$ is a pseudocomplemented distributive lattice. Lakser proved that the subdirectly irreducible algebras are precisely those of the form B^+ for some boolean algebra B. Using this fact he arrives at an interesting corollary, due to K. B. Lee [15]: the lattice of subvarieties of the variety $\mathcal{V}$ of all pseudocomplemented distributive lattices forms a chain, $\mathcal{V}_{-1} \subset \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}$, in which $\mathcal{V}_{-1}$ is the trivial variety, and for each $n \in \omega$, $\mathcal{V}_n$ is the variety generated by the algebra P_n^+, where P_n is the boolean algebra with n atoms. (One can easily check that $\mathcal{V}_0$ is equivalent to the variety of boolean algebras, and $\mathcal{V}_1$ to the variety of Stone algebras.) Since these algebras have reducts which are lattices, each of these varieties is congruence distributive. Therefore, applying Proposition 2.2, for each $n \in \omega$, the variety $\mathcal{V}_n$ is residually small.

Observe that the algebras B^+ are not simple, thus the varieties $\mathcal{V}_n$ are not semi-simple, so Theorem 3.6 does not apply. However, the conclusion of 3.6 is still true, see [7, Theorem 6].

PROPOSITION 5.2. *Let n be a natural number, and $A \in \mathcal{V}_{n\mathrm{SI}}$. The following are equivalent:*

(i) $A \in \mathrm{AMAL}(\mathcal{V}_n)$

(ii) $(A : \mathcal{V}_n) = 1$

(iii) *A is isomorphic to P_0^+, P_1^+ or P_n^+.*

Furthermore, it is also proved in [7, Theorem 1] that $\mathcal{V}$, and hence each of its subvarieties, has the congruence extension property, so a good deal of our work is already done. In fact the only major task is to verify that the result of Proposition 4.4 holds for these varieties.

PROPOSITION 5.3. *Let n be a natural number and $\mathcal{W} = \mathcal{V}_n$ a finitely generated variety of pseudocomplemented distributive lattices. Then every member of* $\mathrm{AMAL}(\mathcal{W})$ *is a subdirect product of members of $\mathcal{W}_{\mathrm{ASI}}$, more specifically, a subdirect product of $\{P_0^+, P_1^+, P_n^+\}$.*

Proof. We reconsider 4.4, checking the results of each numbered step to see that they hold with the hypotheses above.

Step 1: Here we are assuming that $\mathcal{W}$ is congruence distributive and finitely generated. Both assumptions are still valid under our current hypotheses. Of course instead of 3.6 we will apply 5.2. Furthermore, if $T \cong P_0^+$ or P_1^+ then $T \in \mathcal{W}_{\mathrm{ASI}}$ and we are done. Therefore we assume from now on that $T \not\cong P_0^+$ or P_1^+.

Steps 2–3: These arguments apply without any changes.

Step 4: Observe that the only maximal subdirectly irreducible algebra in $\mathcal{V}_n$ is

P_n^+. Thus for each $j \in S$, the algebra N_j constructed will always be isomorphic to P_n^+. The same can be said for the algebras L_0 and L_1 created in step 1, although we can not say anything about the maps α_0 and α_1.

Step 5: Of course here we do not need assumption (1) since $\mathcal{W}$ has the C.E.P.

Step 6: Here a congruence Ψ_1 is found and it is shown that the map μ_1' is an isomorphism of B_1/Ψ_1 with E/Ψ. A new argument is required in our present situation. We have $\mu_1': B_1/\Psi_1 \rightarrowtail E/\Psi$ and $E/\Psi \cong L_0 \cong P_n^+$. One easily checks that any subalgebra of P_n^+ is subdirectly irreducible, thus Ψ_1 must be a completely meet-irreducible congruence of B_1. By 2.1, there is an ultrafilter D on J such that $D_{B_1} \subseteq \Psi_1$ and by 2.2, $B_1/D \cong P_n^+$. Thus B_1/Ψ_1 is a homomorphic image of P_n^+, but every proper homomorphic image of P_n^+ is a boolean algebra. Putting this together, if $D_{B_1} \neq \Psi_1$ then B_1/Ψ_1 would have to be both subdirectly irreducible and boolean, hence it would be isomorphic to P_0^+. But $\phi_1': T \rightarrowtail B_1/\Psi_1$ and by assumption, $T \not\cong P_0^+$. Thus we conclude that $D_{B_1} = \Psi_1$ and μ_1' maps B_1/Ψ_1 isomorphically onto E/Ψ.

Steps 7–8: We have already shown that $D_{B_1} = \Psi_1$. The rest of the argument goes through as before. ■

THEOREM 5.4. *Let $\mathcal{W}$ be a finitely generated variety of pseudocomplemented distributive lattices. Then for $A \in \mathcal{W}$:*

$A \in \mathrm{AMAL}\,(\mathcal{W})$ *if and only if A satisfies property P.*

Proof. The proof is identical to the one used in 4.6 except for the argument showing that $A^\$ \in \mathrm{AMAL}\,(\mathcal{W})$. Let C be an extension of $A^\$$ and $\alpha : A^\$ \to P_n^+$. Let $\Theta = \ker(\alpha)$ and apply the congruence extension property to get $\Psi \in \mathrm{Con}\,(C)$ extending Θ. Θ is completely meet-irreducible, so Ψ can be taken to be completely meet-irreducible. Therefore C/Ψ can be embedded into P_n^+. By the same argument used in Step 6 of the preceeding theorem, $A^\$/\Theta \cong P_0^+$ or P_n^+. Since there is only one embedding of each of these into P_n^+, it must be extended by the map of C/Ψ into P_n^+. Thus the composition $C \twoheadrightarrow C/\Psi \rightarrowtail P_n^+$ extends α. ■

REFERENCES

[1] G. BERGMAN, *Sulle classi filtrali di algebre*, Ann. Univ. Ferrara, (7), *17* (1971), 35–42.

[2] S. BURRIS and H. P. SANKAPPANAVAR, *A Course in Universal Algebra*, Springer-Verlag, New York, 1981.

[3] B. A. DAVEY, *Weak injectivity and congruence extension in conguruence-distributive equational classes*, Canad. J. Math., *29* (1977), 449–459.

[4] E. FRIED, G. GRÄTZER and H. LAKSER, *Amalgamation and weak injectives in the equational class of modular lattices M_n*, Notices Amer. Math. Soc., *18* (1971), 624, #71T–A101.

[5] E. FRIED and E. W. KISS, *Connection between the congruence-lattices and polynomial properties*, Preprint.

[6] G. GRATZER, *Universal Algebra*, 2nd ed., Springer-Verlag, N.Y. 1979.

[7] G. GRÄTZER and H. LAKSER, *The structure of pseudocomplemented distributive lattices, II: Congruence extension and amalgamation*, Trans. Amer. Math. Soc., *156* (1971), 343–358.

[8] G. GRÄTZER, H. LAKSER and B. JÓNSSON, *The amalgamation property in equational classes of lattices*, Pacific J. Math., *45* (1973), 507–524.

[9] G. GÄTZER and E. T. SCHMIDT, *Ideals and congruence relations in lattices*, Acta. Math. Acad. Sci. Hungar., *9* (1958), 137–175.

[10] B. JÓNSSON, *Homogeneous universal relational systems*, Math. Scand., *8* (1960), 137–142.

[11] B. JÓNSSON, *Algebraic extensions of relational systems*, Math. Scand., *11* (1962), 179–205.

[12] B. JÓNSSON, *Algebras whose congruence lattices are distributive*, Math. Scand., *21* (1967), 110–121.

[13] P. KÖHLER and D. PIGOZZI, *Varieties with equationally definable principal congruences*, Algebra Universalis, *11* (1980), 213–219.

[14] H. LAKSER, *The structure of pseudocomplemented distributive lattices. I: Subdirect decomposition*, Trans. Amer. Math. Soc., *156* (1971), 335–342.

[15] K. B. LEE, *Equational classes of distributive pseudocomplmented lattices*, Canad. J. Math., *22* (1970), 881–891.

[16] R. QUACKENBUSH, *Structure theory for equational classes generated by quasi-primal algebras*, Trans. Amer. Math. Soc., *187* (1974), 127–145.

[17] P. RIBENBOIM, *Characterization of the sub-complement in a distributive lattice with last element*, Summa Brasil. Math., *2* (1949), no. 4, 43–49.

[18] A. ROBINSON, *Infinite forcing in model theory*, Proc. Second Scand. Logic Symp., 317–340, North-Holland, Amsterdam.

[19] W. TAYLOR, *Residually small varieties*, Algebra Universalis, *2* (1972), 33–53.

[20] M. YASUHARA, *The amalgamation property, the universal-homogeneous models and the generic models*, Math. Scand., *34* (1974), 5–36.

[21] C. BERGMAN, *The amalgamation class of a discriminator variety is finitely axiomatizable*, Lecture Notes in Math., 1004, Springer-Verlag, Berlin 1983, 1–9.

Iowa State University
Ames, Iowa
U.S.A.