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On the relationship of AP, RS and CEP in congruence modular varieties

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Abstract. A condition is found on a congruence modular variety, guaranteeing that the implication AP & RS $\Rightarrow$ CEP holds. The condition is in terms of the commutator on congruence lattices. In particular, the implication holds for any congruence distributive variety whose free algebra on four generators is finite.

In his survey of equational logic [11], W. Taylor observes that the amalgamation property, residual smallness and the congruence extension property are completely independent properties of a variety, with the possible exception of AP & RS $\Rightarrow$ CEP. By this he means that for each of the other seven "boolean combinations" of the properties AP, RS, CEP, there is a variety exhibiting that combination. In fact, the exemplary variety can always be taken to be congruence distributive. In this paper we demonstrate that the situation is perhaps different for the remaining implication by proving that for a congruence distributive variety whose free algebra on four generators is finite, AP & RS $\Rightarrow$ CEP. Actually we prove more: the implication holds in any congruence modular variety satisfying the commutator conditions ($C2$) and (R) whose free algebra on four generators is finite. That ($C2$) and (R) are necessary conditions for CEP was discovered by E. Kiss [8]. Along the way, we provide a new characterization of CEP in terms of subdirect irreducibles that is an improvement of those currently in the literature.

Our notation will generally follow that of [2]. The congruence lattice of an algebra $\mathbf{A}$ is denoted Con ($\mathbf{A}$), with least and greatest elements 0 and 1. We use lower case Greek letters for congruences. The least non-zero congruence on a subdirectly irreducible algebra $\mathbf{A}$ is called the *monolith*. We reserve the letter μ, or $\mu_{\mathbf{A}}$ if necessary, for that congruence.

An injective homomorphism (embedding) is denoted $\mathbf{A} \rightarrowtail \mathbf{B}$, and a surjective map, $\mathbf{A} \twoheadrightarrow \mathbf{B}$. Suppose f is an embedding of $\mathbf{A}$ into $\mathbf{B}$, and $\beta \in \mathrm{Con}(\mathbf{B})$. Then $f^{-1}(\beta)$ is the congruence $\{(a, b) : (f(a), f(b)) \in \beta\}$ on $\mathbf{A}$. The embedding f induces

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an embedding f/β from $\mathbf{A}/f^{-1}(\beta)$ to $\mathbf{B}/\beta$ given by $f/\beta(x/f^{-1}(\beta)) = f(x)/\beta$, where x/α denotes the equivalence class of x modulo α. If $\alpha \in \operatorname{Con}(\mathbf{A})$, then there is a natural bijection between the interval $\{\beta \in \operatorname{Con}(\mathbf{A}) : \alpha \leqslant \beta \leqslant 1\}$ and $\operatorname{Con}(\mathbf{A}/\alpha)$ that takes β to β/α.

The main theorems make reference to the commutator in congruence modular varieties as developed in [7] (see also [4] or [6] for details.) An algebra $\mathbf{A}$ has the property $(C2)$ if $(\forall \alpha, \beta \in \operatorname{Con}(\mathbf{A}))\ [\alpha, \beta] = \alpha \wedge \beta \wedge [1, 1]$. $\mathbf{A}$ has property (R) if $(\forall \mathbf{B} \leqslant \mathbf{A})[1_{\mathbf{A}}, 1_{\mathbf{A}}] \restriction B = [1_{\mathbf{B}}, 1_{\mathbf{B}}]$. A variety $\mathscr{V}$ has $(C2)$ or (R) if every member has the property. From [8, I.4.2], $\mathscr{V}$ has $(C2)$ if and only if every subdirect irreducible has $(C2)$. Furthermore, $(C2)$ is preserved by finite subdirect products.

Let $\mathbf{C}$ be an algebra satisfying $(C2)$, $\alpha, \eta_1, \eta_2, \ldots, \eta_n$ arbitrary congruences of $\mathbf{C}$, and denote $[1, 1]$ by ξ. By $(C2)$ $(\alpha \vee \eta_1) \wedge (\alpha \vee \eta_2) \wedge \xi = [\alpha \vee \eta_1, \alpha \vee \eta_2] \leqslant \alpha \vee (\eta_1 \wedge \eta_2 \wedge \xi)$. Hence, by modularity, $(\alpha \vee \eta_1) \wedge (\alpha \vee \eta_2) \wedge (\alpha \vee \xi) = \alpha \vee (\eta_1 \wedge \eta_2 \wedge \xi)$. Continuing by induction we obtain:

$$(\alpha \vee \eta_1) \wedge (\alpha \vee \eta_2) \wedge \cdots \wedge (\alpha \vee \eta_n) \wedge (\alpha \vee \xi) = \alpha \vee (\eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_n \wedge \xi).$$

In particular, if $\bigcap \eta_i = 0$ then the right hand side reduces to α. In this case, it is often convenient to think of the left hand side as $(\beta_1 \times \beta_2 \times \cdots \times \beta_n) \wedge (\alpha \vee \xi)$ with β_i a congruence on $\mathbf{C}/\eta_i$.

This paper makes heavy use of the concepts of essential and maximal essential extension, absolute retract, maximal irreducible algebra, and, of course, AP, RS and CEP. The definition and basic properties of these notions can be found in [1] or [12]. In particular, because of the polynomial equivalence of abelian algebras and modules [4, 9.9], every abelian algebra has CEP.

LEMMA 1. *Let* $\mathbf{B}$ *and* $\mathbf{C}$ *be members of a congruence modular variety with* $f: \mathbf{C} \to \mathbf{B}$ *an essential embedding. Suppose there are* $\theta, \psi \in \operatorname{Con}(\mathbf{B})$ *such that* θ *is meet-irreducible,* $\psi \neq 0$ *and* $\theta \wedge \psi = 0$. *Then the induced embedding* f/θ: $C/f^{-1}(\theta) \rightarrowtail \mathbf{B}/\theta$ *is essential.*

Proof. Let β/θ be a congruence on $\mathbf{B}/\theta$ such that $[f/\theta]^{-1}(\beta/\theta) = 0$. Then for the congruence β on $\mathbf{B}$ we have $\beta \geqslant \theta$ and $f^{-1}(\beta) = f^{-1}(\theta)$. Let $\gamma = \beta \wedge (\theta \vee \psi)$. We have $\theta \leqslant \gamma \leqslant \theta \vee \psi$, so $\gamma \vee \psi = \theta \vee \psi$. Furthermore, $\gamma \wedge \psi = 0 = \theta \wedge \psi$ since $f^{-1}(\gamma \wedge \psi) = f^{-1}(\gamma) \wedge f^{-1}(\psi) \leqslant f^{-1}(\beta) \wedge f^{-1}(\psi) = f^{-1}(\theta \wedge \psi) = f^{-1}(\theta \wedge \psi) = 0$ and f is essential. Now, by modularity, $\theta = \gamma$ (else $\{0, \theta, \gamma, \psi, \theta \vee \psi\}$ forms a pentagon.)

Therefore $\theta = \beta \wedge (\theta \vee \psi)$. But by assumption, θ is meet-irreducible. As $\psi \neq 0$, we conclude $\theta = \beta$, hence $\beta/\theta = 0$ in $\operatorname{Con}(\mathbf{B}/\theta)$. $\diamondsuit$

For the remainder of this paper, we work in a fixed congruence modular variety $\mathcal{V}$ such that $\mathcal{V}_{SI}$ satisfies ($C2$) and (R). We need the following technical lemma.

LEMMA 2. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be members of* $\mathcal{V}, f:\mathbf{C} \rightarrowtail \mathbf{D}$ *an embedding,* $\mathbf{C}$ *finite and* α *a meet-irreducible congruence of* $\mathbf{C}$. *If* $\alpha \geqslant [1_{\mathbf{C}}, 1_{\mathbf{C}}]$, *then* α *extends to* $\mathbf{D}$. *If* $\alpha \ngeqslant [1_{\mathbf{C}}, 1_{\mathbf{C}}]$ *then there is a completely meet-irreducible congruence* ν *on* $\mathbf{D}$ *such that* $\alpha \geqslant f^{-1}(\nu)$ *and the induced embedding* f/ν *is essential.*

Proof. Let β be a maximal element of $\{\gamma \in \operatorname{Con}(\mathbf{D}): f^{-1}(\gamma) = 0\}$ (which exists by Zorn's lemma,) $\mathbf{E} = \mathbf{D}/\beta$ and $g = f/\beta$ the induced embedding of $\mathbf{C}$ into $\mathbf{E}$. Then g is essential. If $\mathbf{E}$ is subdirectly irreducible, take $\nu = \beta$, and we are done. So assume $\mathbf{E}$ is not subdirectly irreducible. Write $\mathbf{E}$ as a subdirect product of subdirectly irreducible algebras $(\mathbf{E}_i : i \in I)$. Let η_i be the projection kernel of $\mathbf{E} \twoheadrightarrow \mathbf{E}_i$. Since $\mathbf{C}$ is finite, the set $\{g^{-1}(\eta_i): i \in I\}$ is finite. Let J be a minimal subset of I such that $\bigcap (g^{-1}(\eta_j): j \in J) = 0$. Then $g^{-1}(\bigcap(\eta_j : j \in J)) = 0$, so, since g is essential, $\bigcap (\eta_j : j \in J) = 0$ on $\mathbf{E}$.

Now, J is finite, so by [8, I.3.4], $\mathbf{E}$ satisfies (R). Therefore, if $\alpha \geqslant [1_{\mathbf{C}}, 1_{\mathbf{C}}] = [1_{\mathbf{E}}, 1_{\mathbf{E}}] \restriction C$, then α extends to $\mathbf{E}$ since abelian algebras have CEP. On the other hand, suppose $\alpha \ngeqslant [1, 1]$ on $\mathbf{C}$. Since $\mathcal{V}$ satisfies ($C2$), $\alpha = \bigcap (\alpha \vee g^{-1}(\eta_j): j \in J) \cap (\alpha \vee [1, 1])$. But α is meet-irreducible, so there is $k \in J$ such that $\alpha \geqslant g^{-1}(\eta_k)$. Taking $\theta = \eta_k$ and $\psi = \bigcap (\eta_j : j \in J - \{k\})$ in Lemma 1, the map g/θ is essential. Finally, let ν be the congruence on $\mathbf{D}$ such that $\nu/\beta = \theta$. Then $\mathbf{D}/\nu \cong \mathbf{E}/\theta$, $f^{-1}(\nu) = g^{-1}(\theta)$ and f/ν is essential. $\diamond$

To proceed, we need to strengthen a result from [8] which is, in turn, a generalization of a well-known theorem of Kollár [9] on the CEP. Suppose $\mathbf{C}$ is an essential extension of $\mathbf{B}$ (abbreviated $\mathbf{B} \leqslant_E \mathbf{C}$.) Observe that if $\mathbf{B}$ is subdirectly irreducible, then, so is $\mathbf{C}$. In general the converse if false. However, in the presence of CEP, it is true, and, in fact, the monolith of $\mathbf{C}$ is an extension of the monolith of $\mathbf{B}$. The following theorem yields a converse in a special case. In what follows, $\mathbf{F}(4)$ denotes the $\mathcal{V}$-free algebra on 4 generators.

THEOREM 3. *Let* $\mathbf{F}(4)$ *be finite and suppose that for every* $\mathbf{D} \in \mathcal{V}_{SI}$ *and every finite* $\mathbf{C} \leqslant_E \mathbf{D}$, $\mathbf{C}$ *is subdirectly irreducible and* $\mu_{\mathbf{D}} \restriction C = \mu_{\mathbf{C}}$. *Then* $\mathcal{V}$ *has* CEP.

Proof. We first establish the following claim:

Let $\mathbf{C}$ be a finite, subdirectly irreducible algebra, and $\mathbf{D}$ be an essential extension of $\mathbf{C}$. Then every congruence on $\mathbf{C}$ extends to $\mathbf{D}$.

The proof is by induction on the cardinality of $\mathbf{C}$. If card $(\mathbf{C}) = 2$ then $\mathbf{C}$ is

$$\begin{tikzcd}
\mathrm{C} \arrow[r, "f"] \arrow[d] & \mathrm{D} \arrow[d] \\
\mathrm{C}/\mu_{\mathrm{C}} \arrow[r, "f/\mu_{\mathrm{D}}"] \arrow[d] & \mathrm{D}/\mu_{\mathrm{D}} \arrow[d] \\
\mathrm{C}/f^{-1}(\nu) \arrow[r, "f/\nu"] & \mathrm{D}/\nu
\end{tikzcd}$$

Figure 1

simple and the claim clearly holds. So suppose the claim holds for every algebra of cardinality less than that of C. It suffices to consider a completely meet-irreducible congruence α of $\mathbf{C}$, $\alpha \neq 0$. If $\alpha \geqslant [1_{\mathbf{C}}, 1_{\mathbf{C}}]$, then by Lemma 2, α will extend to $\mathbf{D}$. So, assume $\alpha \not\geqslant [1, 1]$ on $\mathbf{C}$. Let f denote the embedding of $\mathbf{C}$ into $\mathbf{D}$. By the hypothesis to the theorem, $f^{-1}(\mu_{\mathbf{D}}) = \mu_{\mathbf{C}}$. Since $\alpha > 0$, it corresponds to the congruence $\alpha/\mu_{\mathbf{C}}$ on $\mathbf{C}/\mu_{\mathbf{C}}$ which is also completely meet-irreducible. Applying Lemma 2 to the embedding $f/\mu_{\mathbf{D}}: \mathbf{C}/\mu_{\mathbf{C}} \to \mathbf{D}/\mu_{\mathbf{D}}$ and congruence $\alpha/\mu_{\mathbf{C}}$ there is a congruence ν on $\mathbf{D}$ with $\nu \geqslant \mu_{\mathbf{D}}$, $\alpha \geqslant f^{-1}(\nu)$, f/ν essential and $\mathbf{D}/\nu$ subdirectly irreducible (see figure 1). Therefore, by the hypothesis to the theorem, $\mathbf{C}/f^{-1}(\nu)$ is subdirectly irreducible and has cardinality less than that of $\mathbf{C}$, so we can apply the induction hypothesis. The congruence α of $\mathbf{C}$ corresponds to $\alpha/f^{-1}(\nu)$ on $\mathbf{C}/f^{-1}(\nu)$ which extends to β/ν on $\mathbf{D}/\nu$, which in turn corresponds to β on $\mathbf{D}$. Then $f^{-1}(\beta) = \alpha$ as desired.

The theorem now follows easily from the claim and Lemma 2. By Day's well-known result [3], it suffices to consider an arbitrary algebra $\mathbf{B}$ of $\mathscr{V}$, and a subalgebra $\mathbf{A}$ generated by 4 elements. By our assumption on the free algebra, A is finite. Let α be a completely meet-irreducible congruence of $\mathbf{A}$. By Lemma 2, if $\alpha \geqslant [1, 1]$, then α extends to $\mathbf{B}$. If $\alpha \not\geqslant [1, 1]$, then again by Lemma 2, there is a congruence ν on $\mathbf{B}$ such that $\mathbf{B}/\nu$ is subdirectly irreducible and the induced embedding $\mathbf{A}/(\nu \restriction A) \rightarrowtail \mathbf{B}/\nu$ is essential. By the hypothesis to the theorem, $\mathbf{A}/(\nu \restriction A)$ is subdirectly irreducible, hence the claim yields that the projection of α extends to $\mathbf{B}/\nu$, hence α extends to $\mathbf{B}$. $\diamond$

The assumption, in theorem 3, that $\mu_{\mathbf{B}} \restriction A = \mu_{\mathbf{A}}$ can not be dropped, as the following example shows. Let $\mathbf{N}_5$ be the lattice of figure 2 and let * be a new unary operation given by: $0^* = c$, $a^* = b$, $b^* = 1$, $c^* = c$, $1^* = 1$. Let $\mathbf{A} = \langle N_5; \wedge, \vee, {}^*, a \rangle$ of type $(2, 2, 1, 0)$. The only proper subalgebra of $\mathbf{A}$ is subdirectly irreducible with universe $\mathbf{C} = \{a, b, 1\}$. $\mathbf{A}$ has 3 congruences, one of them, μ, partitions A into $\{a, b, 1\}$ and $\{0, c\}$. $\mathbf{C}$ also has 3 congruences, with monolith μ' identifying b with 1. Thus, the variety generated by $\mathbf{A}$ satisfies all the hypotheses of theorem 3 (in fact $\mathscr{V}_{SI} = I\{\mathbf{A}, \mathbf{C}, \mathbf{A}/\mu, \mathbf{C}/\mu'\}$ is closed under subalgebra) except that $\mu \restriction C \neq \mu'$, hence $\mathscr{V}$ does not have CEP.

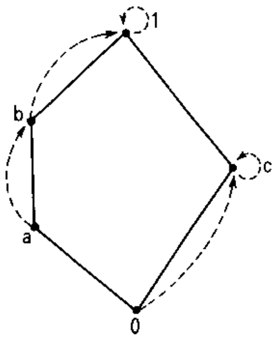

Figure 2

LEMMA 4. *Let* $\mathbf{A}$ *be a non-abelian maximal irreducible of* $\mathscr{V}$ *and* $g:\mathbf{A}\times\mathbf{A}\mapsto\mathbf{H}$, *for some* $\mathbf{H}$ *in* $\mathscr{V}$. *Then there is a congruence* ψ *on* $\mathbf{H}$ *such that* $g^{-1}(\psi)$ *is equal to a coordinate projection kernel on* $\mathbf{A}\times\mathbf{A}$.

Proof. Let $d:\mathbf{A}\mapsto\mathbf{A}\times\mathbf{A}$ be the diagonal embedding. As $\mathbf{A}$ is a maximal irreducible, it is an absolute retract of $\mathscr{V}$. Therefore, there is a map $r:\mathbf{H}\twoheadrightarrow\mathbf{A}$ such that $r\circ g\circ d = id_A$. Let $r' = r\circ g$. Then $r'\circ d = id_A$, so r' is a surjective map from $\mathbf{A}\times\mathbf{A}$ to $\mathbf{A}$. Let $\delta = \ker(r')$, Let η_0 and η_1 denote the coordinate projection kernels on $\mathbf{A}\times\mathbf{A}$. As $(\mathbf{A}\times\mathbf{A})/\delta\cong\mathbf{A}$, δ is meet-irreducible and $\delta\not\geqslant[1, 1]$. By $(C2)$, $\delta = (\delta\vee\eta_0)\wedge(\delta\vee\eta_1)\wedge(\delta\vee[1, 1])$, hence $\delta\geqslant\eta_0$ or $\delta\geqslant\eta_1$. Suppose $\delta\geqslant\eta_0$. Then for every $a, b\in A$, $(a, a)\delta(a, b)$. Therefore $r'(a, b) = r'(a, a) = r'\circ d(a) = a$, in other words, $\delta = \eta_0$. Similarly, $\delta\geqslant\eta_1$ implies $\delta = \eta_1$. Now we take $\psi = \ker(r)$. $\diamond$

LEMMA 5. *Assume* $\mathscr{V}$ *is residually small. Suppose there are non-trivial algebras* $\mathbf{B}_0$ *and* $\mathbf{B}_1$ *in* $\mathscr{V}$, $\mathbf{A}\in\mathscr{V}_{SI}$ *and* $\mathbf{A}$ *is an essential extension of* $\mathbf{B}_0\times\mathbf{B}_1$. *Then* $\mathscr{V}$ *does not have* AP.

Proof. Since $\mathscr{V}$ is residually small, we may assume that $\mathbf{A}$ is a maximal irreducible algebra of $\mathscr{V}$ [12, Theorem 1.2, 1.8]. $\mathbf{A}$ fails to have CEP (since $\mathbf{B}_0\times\mathbf{B}_1$ is not subdirectly irreducible) so is certainly non-abelian. Let $\mathbf{C} = \mathbf{B}_0\times\mathbf{B}_1\times\mathbf{B}_0\times\mathbf{B}_1$, $\mathbf{D} = \mathbf{E} = \mathbf{A}\times\mathbf{A}$. Denote the inclusion of $\mathbf{B}_0\times\mathbf{B}_1$ into $\mathbf{A}$ by i and let

$$f:\mathbf{C}\to\mathbf{D}\quad\text{by}\quad f(c) = (i(c_0, c_1), i(c_2, c_3))$$

$$f:\mathbf{C}\to\mathbf{E}\quad\text{by}\quad g(c) = (i(c_0, c_3), i(c_2, c_1)).$$

We claim that $(\mathbf{C}, \mathbf{D}, \mathbf{E}, \mathrm{f}, \mathrm{g})$ can not be amalgamated. For suppose it is, by $(\mathbf{H}, f', g')$. Since $\mathbf{A}$ is a non-abelian maximal irreducible, by Lemma 4, there is $\psi\in\mathrm{Con}(\mathbf{H})$ with $f^{-1}(\psi)$ equal to, say, η_0 on $\mathbf{D}$.

Let $\alpha = f^{-1}(\eta_0) = (g' \circ g)^{-1}(\psi)$ on $\mathbf{C}$. Clearly, for $x, y \in \mathbf{D}$,

$$x \equiv y (\operatorname{mod} \alpha) \Leftrightarrow x_0 = y_0 \quad \text{and} \quad x_1 = y_1. \tag{*}$$

Let $\beta = g'^{-1}(\psi)$ so $g^{-1}(\beta) = \alpha$. By $(C2)$, $\beta = (\beta_0 \times \beta_1) \wedge (\beta \vee [1, 1])$. Since $[1_{\mathbf{E}}, 1_{\mathbf{E}}] = [1_{\mathbf{A}}, 1_{\mathbf{A}}] \times [1_{\mathbf{A}}, 1_{\mathbf{A}}]$, we have $\beta \geqslant (\beta_0 \times \beta_1) \wedge [1, 1] = (\beta_0 \wedge [1, 1]) \times (\beta_1 \wedge [1, 1])$.

By assumption, i is an essential extension, so $i^{-1}(\mu_{\mathbf{A}}) \neq 0$. Then there are distinct pairs (c_0, c_1) and (d_0, d_1) with $i(c_0, c_1) \equiv i(d_0, d_1) \ (\operatorname{mod} \mu_{\mathbf{A}})$. Assume $c_0 \neq d_0$, the other case being analogous. We must have $\beta_0 = 0$ on A. For, if not, then $\beta_0 \geqslant \mu$. Also, $\mathbf{A}$ is non-abelian, so $[1, 1] \geqslant \mu$. Therefore $\beta \geqslant (\beta_0 \wedge [1, 1]) \times (\beta_1 \wedge [1, 1]) \geqslant \mu \times 0$. Consider the elements $c = (c_0, c_1, c_0, c_1)$ and $d = (d_0, c_1, c_0, d_1)$ of C. Clearly, $g(c) \equiv g(d) (\operatorname{mod} \mu \times 0)$, so $c \equiv d (\operatorname{mod} \alpha)$. But $c_0 \neq d_0$ contradicting (*).

Therefore $\beta = (0 \times \beta_1) \wedge (\beta \vee [1, 1]) \leqslant 0 \times \beta_1$. Hence $x \equiv y (\operatorname{mod} \alpha)$ implies $i(x_0, x_3) = i(y_0, y_3)$ which in turn implies $x_3 = y_3$, contradicting (*). $\diamondsuit$

We can now prove the main theorem.

THEOREM 6. *Let $\mathcal{V}$ be a modular variety such that $\mathbf{F}_{\mathcal{V}}(4)$ is finite and $\mathcal{V}_{SI}$ satisfies* $(C2)$ *and* (R). *If $\mathcal{V}$ has* AP & RS *then $\mathcal{V}$ has* CEP.

Proof. We wish to apply Theorem 3. Let $\mathbf{B} \in \mathcal{V}_{SI}$ and let $\mathbf{A}$ be a finite algebra such that $f: \mathbf{A} \to \mathbf{B}$ is an essential embedding. Define $\xi = f^{-1}(\mu_{\mathbf{B}})$. We must show that ξ is the monolith of $\mathbf{A}$. If $\mathbf{B}$ is abelian, this is immediate. So we may assume $\mathbf{B}$ is non-abelian. It suffices to consider a completely meet-irreducible congruence α on $\mathbf{A}$, with $\alpha \ngeqslant \xi$ and show that $\alpha = 0$. Observe that $[1_{\mathbf{B}}, 1_{\mathbf{B}}] \geqslant \mu_{\mathbf{B}}$ and $\mathbf{B}$ satisfies (R), so $[1_{\mathbf{A}}, 1_{\mathbf{A}}] \geqslant \xi$. Thus $\alpha \ngeqslant [1_{\mathbf{A}}, 1_{\mathbf{A}}]$.

Let $g: \mathbf{A} \twoheadrightarrow \mathbf{A}/\alpha$ be the canonical projection. By the amalgamation property, the diagram $(\mathbf{A}, \mathbf{B}, \mathbf{B} \times (\mathbf{A}/\alpha), f, f \times g)$ can be amalgamated, say by $(\mathbf{E}, s, t)$. Let σ be a maximal element of $\{\theta \in \operatorname{Con}(\mathbf{E}) : t^{-1}(\theta) = 0 \text{ on } \mathbf{B} \times (\mathbf{A}/\alpha)\}$, and let $q: \mathbf{E} \twoheadrightarrow \mathbf{E}/\sigma$. Then $(t \circ (f \times g))^{-1}(\alpha) = 0 = (s \circ f)^{-1}(\sigma)$, so $s^{-1}(\sigma) = 0$ (on $\mathbf{B}$) since f is an essential extension. In other words, both $q \circ s$ and $q \circ t$ are embeddings and $q \circ t$ is an essential extension. Writing $(\mathbf{D}, f', h)$ in place of $(\mathbf{E}/\alpha, q \circ s, q \circ t)$ we have: $(\mathbf{D}, f', h)$ amalgamates $(\mathbf{A}, \mathbf{B}, \mathbf{B} \times (\mathbf{A}/\alpha), f, f \times g)$ and h is essential. (See figure 3).

$$\begin{array}{ccccc} \mathbf{A} & \xrightarrow{f} & \mathbf{B} & & \\ {\scriptstyle f \times g}\downarrow & & \downarrow{\scriptstyle f'} & & \\ B \times (\mathbf{A}/\alpha) & \xrightarrow[h]{} & \mathbf{D} & \xrightarrow{p_i} & \mathbf{D}_i \end{array}$$

Figure 3

Write $\mathbf{D}$ as a subdirect product of subdirectly irreducible algebras, $(\mathbf{D}_i : i \in I)$, and let p_i be the projection of $\mathbf{D}$ onto $\mathbf{D}_i$ for $i \in I$. By $(C2)$, $\ker(p_i \circ h) = (\delta_i \times \gamma_i) \wedge \beta_i$, with $\delta_i \in \operatorname{Con}(\mathbf{B})$, $\gamma_i \in \operatorname{Con}(\mathbf{A}/\alpha)$ and $\beta_i \geq [1, 1]$ on $\mathbf{B} \times \mathbf{A}/\alpha$. As $[1, 1]$ is a product congruence, we may write $(\delta_i \times \gamma_i) \wedge \beta_i \geq (\delta_i \wedge [1, 1]) \times (\gamma_i \wedge [1, 1])$. Now $0 = \bigcap \ker(p_i \circ h) \geq \bigcap (\delta_i \wedge [1, 1]) \times \bigcap (\gamma_i \wedge [1, 1])$, every intersection being over the index set I. Both $\mathbf{B}$ and $\mathbf{A}/\alpha$ are subdirectly irreducible and non-abelian, so there are indices j and k in I such that $\delta_j = 0$ on $\mathbf{B}$ and $\gamma_k = 0$ on $\mathbf{A}/\alpha$. In particular, if η denotes the kernel of the projection $\mathbf{B} \times \mathbf{A}/\alpha \twoheadrightarrow \mathbf{A}/\alpha$, then $\ker(p_k \circ h) \leq \eta$, so by modularity, $\ker(p_k \circ h)$ is a product congruence of the form $\delta_k \times 0$, with $\delta_k \in \operatorname{Con}(\mathbf{B})$.

Then $\ker(p_k \circ h \circ f \times g) = (f \times g)^{-1}(\delta_k \times 0) = f^{-1}(\delta_k) \wedge \alpha$. From this we see that $\ker(p_k \circ f') = 0$ on $\mathbf{B}$. For otherwise, $\ker(p_k \circ f') \geq \mu$, hence $\ker(p_k \circ f' \circ f) \geq \xi$. But figure 3 commutes, so $\ker(p_k \circ f' \circ f) = \ker(p_k \circ h \circ f \times g) = f^{-1}(\delta_k) \wedge \alpha \leq \alpha$ which is impossible since $\alpha \ngeq \xi$. Thus, $p_k \circ f'$ is an embedding of $\mathbf{B}$ into $\mathbf{D}_k$, and $f^{-1}(\delta_k) \wedge \alpha = 0$.

On the other hand, consider the map $(p_j \times p_k) : \mathbf{D} \to \mathbf{D}_j \times \mathbf{D}_k$. We have $\ker(p_j \times p_k \circ h) = \ker(p_j \circ h) \wedge \ker(p_k \circ h) = (0 \times \gamma_j) \wedge \beta_j \wedge (\delta_k \times 0) = 0$. Since h is essential, we conclude that $\ker(p_j \times p_k) = 0$, thus $\mathbf{D}$ is a subdirect product of $\mathbf{D}_j$ and $\mathbf{D}_k$.

Suppose that $\ker(p_j) = 0$. Then (since p_j is also surjective) $\mathbf{D} \cong \mathbf{D}_j$ is subdirectly irreducible. But then h is an essential embedding of $\mathbf{B} \times \mathbf{A}/\alpha$ into $\mathbf{D}_j$, contradicting Lemma 5.

Therefore $\ker(p_j) \neq 0$. By Lemma 1, the induced map $h/\ker(p_k) : (\mathbf{B}/\delta_k) \times (\mathbf{A}/\alpha) \rightarrowtail \mathbf{D}_k$ is essential. However, this is again a contradiction of Lemma 5, unless $\delta_k = 1$ on $\mathbf{B}$. In that case, $0 = f^{-1}(\delta_k) \wedge \alpha = f^{-1}(1) \wedge \alpha = \alpha$ as desired. ◇

A congruence modular variety is congruence distributive if and only if $[\alpha, \beta] = \alpha \wedge \beta$ for any two congruences on any algebra in the variety. Thus, a congruence distributive variety always satisfies $(C2)$ and (R).

COROLLARY 7. *Let $\mathcal{V}$ be a congruence distributive variety such that $\mathbf{F}_{\mathcal{V}}(4)$ is finite. If $\mathcal{V}$ has* AP *and* RS, *then $\mathcal{V}$ has* CEP.

For any variety, the conjunction of AP, RS and CEP is equivalent to injective completeness. In Kollár [9], there is a characterization of injective completeness in terms of subdirectly irreducible algebras, for finitely generated, congruence distributive varieties. As such varieties always have RS, this condition is equivalent to AP.

COROLLARY 8. *Let $\mathcal{V}$ be a finitely generated, congruence distributive variety. The following are equivalent:*

i) *$\mathcal{V}$ is injectively complete*
ii) *$\mathcal{V}$ has* AP
iii) *Every maximal subdirectly irreducible algebra is $\mathcal{K}$-injective, and every retract of a maximal subdirectly irreducible algebra is the product of subdirectly irreducible algebras that are $\mathcal{K}$-injective.*

Here $\mathcal{K}$ is the closure of $\mathcal{V}_{SI}$ under homomorphic images and subalgebras.

In a forthcoming paper, we will prove that the assumption that $\mathcal{V}$ have (R) can be dropped. It is our conjecture that $(C2)$ can be dropped as well (that is, AP & RS $\Rightarrow (C2)$.) At this time we have only a lack of counter-examples to support that opinion. However, it is known that residual smallness alone implies another commutator identity, $(C1)$, that is weaker than $(C2)$, see [5, theorem 7] for details. It is also possible that either the requirement that $\mathbf{F}_{\mathcal{V}}(4)$ be finite or that $\mathcal{V}$ be residually small can be omitted from the theorem – but not both (consider that variety of all lattices.)

J. D. H. Smith has observed that the condition on $\mathbf{F}_{\mathcal{V}}(4)$ in theorems 3 and 6 can be weakened to: Con $(\mathbf{F}_{\mathcal{V}}(4))$ has finite height.

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