# Compressional-wave propagation in porous media saturated with two fluids

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## ABSTRACT

A theory of porous media with voids filled by two immiscible fluids should comply with Laplace's law of capillary pressure at the pore level. For the convenience of practical use, it ideally should also involve only the generic elastic coefficients of the mineral grains and pore-filling fluids and avoid the use of "bulk" coefficients of the dry solid frame, whose measurement involves idealized experiments, or poorly understood "phase-coupling" coefficients. It should be reconcilable with the body of empirical data as well. Such a theory and the resulting propagation of the conventional compressional wave can be deduced from principles of linear elasticity. Although the interfacial tension between the fluids is rigorously included, it turns out to have no effect on the velocity of seismic waves. To be reconcilable with observational data, the theory needs to be modified to (1) postulate a power-law effect of the volume fraction of the solid and fluid phases on the reduction of their elastic moduli contributing to the aggregate value and (2) honor the additivity of fluid compressibilities versus bulk moduli to form the effective bulk modulus of the fluids. Empirical calibration of the constants of the power law is necessary to make the theory applicable to a specific class of rock. Constructed this way for well-cemented rocks such as sandstones or limestones, the theory agrees well with the empirical data describing (1) the bulk modulus of the dry solid frame, (2) the bulk modulus of the solid frame filled with one fluid, both as functions of porosity; (3) the data on wave velocity in such rocks filled with one fluid, as a function of porosity, and (4) the measurements of wave velocity in such rocks filled with air and water, as a function of water saturation.

## INTRODUCTION

The theory of elastic-wave propagation in a porous medium filled with one fluid dates back to the seminal works by Frenkel (1944) and Biot (1956a, 1956b). Biot's contribution is that he deduced the theory by generalizing the first principles of classic linear elasticity. The low-frequency theories of Frenkel and Biot are conceptually equivalent, except that Frenkel does not numerically investigate the properties of the propagating waves. Both studies formally predicted the existence of two types of compressional waves, a consequence of two separate coupled equations of motions applied individually to the liquid and solid phases. Since these works, as their natural extension, various authors have attempted to generalize the theory to the case of a porous medium filled with two immiscible fluids in contact with each other, one wetting and one nonwetting. Such a scenario would, for example, represent an oil reservoir, in which a typical wetting fluid would be brine or water, and the nonwetting one would be oil or natural gas (air could be supplemented for gas). A fundamental difference between the single-fluid case and the one of two fluids filling the pores is that the latter scenario involves an interface between the liquids and therefore has to account for the interfacial tension and capillary phenomena.

Brutsaert (1964) historically is the first to directly extend Biot's theory to the case of two fluids, although the author chooses not to include the surface-tension forces (Brutsaert, 1964, p. 244). The result of the straightforward, first-principle-based extension was the three coupled equations of motion applied separately to each phase, which led to the Biot-type prediction of the existence of three compressional waves. On the other hand, Nikolaevskij (1990, section 5.3, the original Russian edition of 1984) apparently is the first to write the general system of equations of mechanics of a multiphase porous medium.

Subsequent works on the theory of porous media with two fluids (Garg and Nayfeh, 1986; Santos et al., 1990a, 1990b; Tuncay and

Corapcioglu, 1996, 1997; Wei and Muraleetharan, 2002; Lo et al., 2005; Lu et al., 2007) have shared a conceptually common approach, differing in the details of implementation. The method generally consisted in developing stress-strain relations for the constitutive phases and then using them in a separate Newton's equation of motion (also known as the momentum-balance equation) for all three phases as part of the standard system of equations of continuum mechanics. Following Frenkel and Biot, this leads to dispersion relations for the propagation of harmonic waves. From the latter, the existence of three compressional waves was deduced in all of the studies. In this approach, the "third" compressional wave arises from the presence of the pressure difference between the two fluids (the capillary pressure). The wave is conceptually equivalent to Biot's "slow" ("second") wave and possesses a similar diffusive character (that is, low speed and high attenuation), making it difficult for practical observation. The existence of the third P-wave (P3) is, to the author's knowledge, still purely hypothetical.

Unlike Brutsaert (1964), the subsequent papers (Garg and Nayfeh, 1986; Santos et al., 1990a, 1990b; Tuncay and Corapcioglu, 1996, 1997; Wei and Muraleetharan, 2002; Lo et al., 2005; Lu et al., 2007) quantitatively incorporate the existence of capillary pressure into their models. None of them do it, though, through the inclusion of the basic physical law of capillarity (Laplace's law of capillary pressure) applied at the pore level, but rather through the "capillary-pressure-versus-saturation" relationship. The latter is a poorly constrained empirical dependence. As a result, the fundamental physical property of interfacial tension does not appear, as a parameter, in any of the theories. This makes it difficult to assess the contribution of interfacial tension to the properties of propagating waves. Another common trait of the prior theories is the employment in all of them, in addition to the elastic constants of the solid and liquid constituents, of the constants of the bulk (or "dry") material, that is, the Gassmann-type elastic parameters of the solid frame with empty pores. The determination of the bulk constants requires idealized experiments, which makes their values seldom (if ever) available, unlike the generic constants of the constituents (mineral grains, fluids) that can be readily found. The reliance on the bulk constants of the solid matrix makes the practical application of these theories difficult.

Generally, the theoretical inferences of Garg and Nayfeh (1986), Tuncay and Corapcioglu (1996, 1997), Wei and Muraleetharan (2002), and Lo et al. (2005) regarding the properties of the compressional waves of the first, second, and third kind are mutually consistent, while those of Santos et al. (1990a, 1990b) and Lu et al. (2007) stand alone. For example, as pointed out by Lo et al. (2005, p. 15), the phase velocity of the P3 wave should be expected to vanish in the limiting case of zero saturation of one liquid phase, in the way of transition to the classic one-fluid case in which only the P1 and P2 waves exist. However, Santos et al. (1990a, Figure 3) show a nonzero velocity of the P3 wave at zero saturation of the nonwetting phase. Other inconsistencies can be noted. Tuncay and Corapcioglu (1996, Figure 3), Wei and Muraleetharan (2002, Figure 2c), and Lo et al. (2005, Figure 3) all predict a minimum in the velocity of the P2 wave at high wetting-phase saturation; this minimum is absent from the respective graph of Santos et al. (1990a, Figure 2). The former three studies deduce a monotonous decline in the P2 velocity from low to high wetting-phase saturation until the minimum is reached; on the other hand, Santos et al. (1990a, Figure 2) and Lu et al. (2007, Figure 6c) show a

monotonous increase. Likewise, Tuncay and Corapcioglu (1996, Figure 4), Wei and Muraleetharan (2002, Figure 2d), and Lo et al. (2005, Figure 5) find a maximum in the velocity of the P3 wave at high wetting-phase saturation, which is not seen in the graph of Santos et al. (1990a, Figure 3) either. The former three studies depict a monotonous increase in the P3 velocity from low to high wetting-phase saturation until the maximum is reached, while Santos et al. (1990a, Figure 3) and Lu et al. (2007, Figure 6d) again show the opposite behavior.

Inconsistencies can be found in the patterns of attenuation as well. Tuncay and Corapcioglu (1996, Figure 11), Wei and Muraleetharan (2002, Figure 3c), and Lo et al. (2005, Figure 4a and 4b) all find a steady rise in the attenuation coefficient of the P2 wave until it reaches a maximum at high wetting-phase saturations. Lu et al. (2007, Figure 7c) deduce a steady decline. In the same vein, Tuncay and Corapcioglu (1996, Figure 12), Wei and Muraleetharan (2002, Figure 3d), and Lo et al. (2005, Figure 6a and 6b) predict a decrease in the attenuation of the P3 wave to a broad minimum from low to high wetting-fluid saturations, which contradicts Lu et al. (2007, Figure 7d) who show an increase. Mismatches between the results of Wei and Muraleetharan (2002) and Lu et al. (2007) exist despite the fact that both studies are founded on a very similar formalism of the mixture theory (see, e.g., their key equations 42–43 and 16a–16b, respectively).

Still further, Santos et al. (1990a, Figures 10 and 11) derive a constant level in the attenuation of the P2 and P3 waves at low frequencies and an unrealistic reduction of attenuation to zero at high frequencies; the latter causes the authors themselves to question the correctness of their theory at high frequencies (Santos et al., 1990a, p. 1436). This overall attenuation pattern contradicts Garg and Nayfeh (1986, Figures 3 and 4) and Lu et al. (2007, Figure 7c and 7d), who conversely (and more realistically) predict a rise in the attenuation of P2 and P3 waves as the frequency increases until it levels off at high frequencies.

It should be noted that none of the authors attempt to verify their theories against the body of empirical data. An exception, in a limited sense, is the work by Tuncay and Corapcioglu (1996, Figure 14) and Wei and Muraleetharan (2002, Figure 2a and 2b). Both studies compare the theoretical predictions for the phase velocities of the compressional wave of the first kind and the shear wave, as a function of wetting-phase saturation, with the experimental data obtained for Massilon sandstone by Murphy (1984, Figure 1). The comparisons nonetheless cannot be considered independent verifications of the theories. Both used the values of the bulk moduli of the dry frame provided by Murphy (1984, Table 5); however, Murphy (1984, p. 11,550) specifically adjusts these parameters to fit the data. The theoretical "fit" to the data was therefore not surprising.

Three questions arise from this overview of the existing studies. First, the theory of wave propagation in a porous medium saturated with two fluids, explicitly complying with Laplace's law of capillary pressure at the pore level, is lacking. Such a theory is needed, for example, to ascertain whether the interfacial tension between the fluids can affect the seismic-wave velocities to the extent that could be detectable by the seismic method. This would make the distinction between porous volumes filled with one fluid and ones filled with two fluids possible in exploration. Second, a preferable theory would contain the generic elastic constants of the solid and liquid constituents only and would avoid the use of the bulk parameters, which require idealized experiments for their determination and are

rarely available. Third, the theory should be reconcilable with the body of empirical data.

We address these questions in the present study. We first derive a theory from the first principles that is made to specifically comply with the law of capillary pressure. We then deduce the wave-propagation velocity from it. Finally, we test the predicted wave velocities, as a function of porosity and fluid saturation, against common empirical relationships used in seismic exploration and against laboratory data. Fitting the data helps remove ambiguities in the formulation of a single velocity expression.

## THEORY

### Equation of motion

Following the classic linear elasticity and the study by Biot (1955), we start with the expression for the strain energy per unit volume, or $V$ in Biot's notation. Because the components of the stress tensor are the first-order partial derivatives of $V$ with respect to the components of the strain tensor (Landau and Lifshitz, 1959, equation 3.6), in order for the stress-strain relation to remain linear, the expression for $V$ can only contain quadratic terms in the strain tensor. In addition, such an expression should remain invariant with respect to coordinate transformation and thus can only contain the invariants of the strain tensor. The invariants are listed by Love (1944, section 13, equation 13). The most general quadratic expression formed from these invariants in the isotropic case is the direct generalization of Love (1944, section 69, equation 17) and Biot (1955, equation 4.1) to the presence of the solid and two fluids:

$$
\begin{aligned}
2V = {}& (\lambda + 2\mu)e^2 + \mu(e_{yz}^2 + e_{zx}^2 + e_{xy}^2 - 4e_{yy}e_{zz} \\
& - 4e_{zz}e_{xx} - 4e_{xx}e_{yy}) + 2Q_1 e\varepsilon_1 + 2Q_2 e\varepsilon_2 \\
& + 2R_1\varepsilon_1\varepsilon_2 + R_2\varepsilon_1^2 + R_3\varepsilon_2^2,
\end{aligned}
\tag{1}
$$

where $\lambda$ and $\mu$ are Lamé's coefficients of the solid and $Q_1$, $Q_2$, $R_1$, $R_2$, $R_3$ are the elastic coefficients required by the presence of two fluids; the meaning of the latter set of coefficients is yet to be clarified. We have also used the notation

$$
e \equiv e_{xx} + e_{yy} + e_{zz} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \operatorname{div}\vec{u},
$$

$$
\varepsilon_1 \equiv \varepsilon_{1xx} + \varepsilon_{1yy} + \varepsilon_{1zz} = \frac{\partial U_{1x}}{\partial x} + \frac{\partial U_{1y}}{\partial y} + \frac{\partial U_{1z}}{\partial z} = \operatorname{div}\overrightarrow{U_1},
$$

$$
\varepsilon_2 \equiv \varepsilon_{2xx} + \varepsilon_{2yy} + \varepsilon_{2zz} = \frac{\partial U_{2x}}{\partial x} + \frac{\partial U_{2y}}{\partial y} + \frac{\partial U_{2z}}{\partial z} = div\,\overrightarrow{U_2}, \tag{2}
$$

where $e_{ij}$, $\varepsilon_{1ij}$, and $\varepsilon_{2ij}$ are the strain tensors in the solid, non-wetting, and wetting-fluid phases, respectively, and $\vec{u} = \{u_x, u_y, u_z\}$, $\overrightarrow{U_1} = \{U_{1x}, U_{1y}, U_{1z}\}$, $\overrightarrow{U_2} = \{U_{2x}, U_{2y}, U_{2z}\}$ are the particle-displacement vectors in the solid, nonwetting, and wetting phases, respectively. In representing the strain energy in the form of equation 1, we make the assumption of spatially uniform distribution of fluid saturation and porosity.

Note that we retain the original meaning of $\lambda$ and $\mu$ in equation 1 as the elastic parameters of the solid phase, as they still appear at the terms describing the deformation of the pure solid. Conversely, Biot

(1955, equation 4.1), followed by Biot and Willis (1957, equation 21) chose to completely redefine these parameters.

With this difference in the meaning of Lamé's coefficients, equation 1 for the strain energy is the same as that used by Brutsaert (1964) (the latter author's equation for $2V$ on p. 246, with the appropriate change in notation), with one exception. The expressions for the energy in the solid as a function of strain, given by Love (1944, section 69, equation 17, which is the same as the sum of the first two terms in the right-hand side of our equation 1) and Landau and Lifshitz (1959, equation 4.1), are slightly different. Namely, Landau and Lifshitz' expression contains a factor of four at the $e_{yz}^2$, $e_{zx}^2$, and $e_{xy}^2$ terms. Biot (1955, equation 4.1) uses the strain energy in Love's form, while the form of Brutsaert is the same as that of Landau and Lifshitz. The difference is apparently due to the fact that Landau and Lifshitz (1959, equation 1.5) define the nondiagonal components of the strain tensor as $e_{xy} = (1/2)(\partial u_x/\partial y + \partial u_y/\partial x)$, etc., while Love (1944, section 8, equation 2), followed by Biot (1955, equations 2.5), do not use the factor of 1/2 in the definition. Both forms of strain energy lead to the same formulation of Hooke's law.

We denote $\sigma_{ij}$, $\sigma_{1ij}$, $\sigma_{2ij}$ as the stress tensors in the solid, nonwetting, and wetting phases, respectively. The stress-tensor components in the solid are obtained from equation 1 as

$$
\sigma_{xx} = \frac{\partial V}{\partial e_{xx}} = 2\mu e_{xx} + \lambda e + Q_1\varepsilon_1 + Q_2\varepsilon_2,
$$

$$
\sigma_{yy} = \frac{\partial V}{\partial e_{yy}} = 2\mu e_{yy} + \lambda e + Q_1\varepsilon_1 + Q_2\varepsilon_2,
$$

$$
\sigma_{zz} = \frac{\partial V}{\partial e_{zz}} = 2\mu e_{zz} + \lambda e + Q_1\varepsilon_1 + Q_2\varepsilon_2,
$$

$$
\sigma_{xy} = \frac{\partial V}{\partial e_{xy}} = \mu e_{xy},
$$

$$
\sigma_{yz} = \frac{\partial V}{\partial e_{yz}} = \mu e_{yz},
$$

$$
\sigma_{zx} = \frac{\partial V}{\partial e_{zx}} = \mu e_{zx}. \tag{3}
$$

For the nonwetting fluid, we write $\sigma_{1xx} = \partial V/\partial \varepsilon_{1xx} = (\partial V/\partial \varepsilon_1)(\partial \varepsilon_1/\partial \varepsilon_{1xx}) = \partial V/\partial \varepsilon_1$. We then introduce $\sigma_1 \equiv \sigma_{1xx} = \sigma_{1yy} = \sigma_{1zz} = \partial V/\partial \varepsilon_1$. Similarly, for the wetting fluid, $\sigma_2 = \partial V/\partial \varepsilon_2$. The quantities $\sigma_1$ and $\sigma_2$ are the stresses applied to the fluids. From this, by differentiating equation 1, we obtain

$$
\sigma_1 = Q_1 e + R_1\varepsilon_2 + R_2\varepsilon_1,
$$

$$
\sigma_2 = Q_2 e + R_1\varepsilon_1 + R_3\varepsilon_2. \tag{4}
$$

Equations 3 and 4 represent the generalization of the stress-strain relations of Biot (1955, equation 4.3) to the case of two fluids.

We now need to combine the stress-tensor components for all phases into a generalized (bulk) stress tensor $\tau_{ij}$ of the porous medium. The solid and fluid stress tensors, formally introduced without regard to the presence of other phases, must then be apportioned according to the porosity $f$:

$$\tau_{ij} = \begin{pmatrix} (1-f)\sigma_{xx}+fS_1\sigma_1+fS_2\sigma_2 & (1-f)\sigma_{xy} & (1-f)\sigma_{xz} \\ (1-f)\sigma_{yx} & (1-f)\sigma_{yy}+fS_1\sigma_1+fS_2\sigma_2 & (1-f)\sigma_{yz} \\ (1-f)\sigma_{zx} & (1-f)\sigma_{zy} & (1-f)\sigma_{zz}+fS_1\sigma_1+fS_2\sigma_2 \end{pmatrix},$$
(5)

where $S_1$ and $S_2$ are the volume saturations of the nonwetting and wetting fluids, respectively ($S_1 + S_2 = 1$). The symmetry $\tau_{ij} = \tau_{ji}$ is assumed. The way $\tau_{ij}$ is defined in equation 5 is the same as, for example, in equation 96 of Morency and Tromp (2008) but departs from the respective equation 2.3 of Biot (1955) in our apportioning the stresses in the constituents according to the porosity. The reason for this departure is that Biot opts to introduce the constituent stresses that, in his words, "do not have the conventional significance" (Biot, 1955, p. 182), while we prefer, for the ease of interpretation, to retain the conventional meaning. Then equation 5 generalizes the original definition of stress as force per unit surface of a single phase to a multiphase compound. Our $\sigma_1$ and $\sigma_2$ are the conventional negative fluid pressures ($p$ in the notation of Biot, 1955, equation 2.4). The way of incorporating porosity as in equation 5 will make it possible, as discussed in a subsequent section, for the porosity to naturally and directly enter the equations of motion, instead of forcing it into the elastic coefficients, as was done, for example, by Biot and Willis (1957, equation 21).

The $i$th component of the force acting on a unit-volume element of the porous medium is calculated using Cauchy's formula, $\partial\tau_{ij}/\partial x_j \equiv \partial\tau_{ix}/\partial x + \partial\tau_{iy}/\partial y + \partial\tau_{iz}/\partial z$. It can be explicitly written using $\tau_{ij}$ from equation 5, the stress-tensor components $\sigma_{ij}$ in the solid from equation 3, the stresses $\sigma_1$ and $\sigma_2$ in the fluids from equation 4, the quantities $e, \varepsilon_1, \varepsilon_2$ from equation 2, and the strain tensor in Love's form $e_{xx} = \partial u_x/\partial x$, $e_{xy} = \partial u_x/\partial y + \partial u_y/\partial x$, etc. To form Newton's equation of motion of the element, this force component should be equated to the mass multiplied by the $i$th component of the acceleration $\ddot{\vec{u}}_{el}$ of the center-of-mass of the element. Omitting the intervening algebra, we arrive at the equation of motion in the vector form

$$[(1-f)\rho_s + f(S_1\rho_1 + S_2\rho_2)]\ddot{\vec{u}}_{el}$$
$$= (1-f)\mu\Delta\vec{u} + [(1-f)(\lambda+\mu)$$
$$+ f(S_1Q_1 + S_2Q_2)]\text{grad div}\vec{u}$$
$$+ [(1-f)Q_1 + f(S_1R_2 + S_2R_1)]\text{grad div}\overrightarrow{U_1}$$
$$+ [(1-f)Q_2 + f(S_1R_1 + S_2R_3)]\text{grad div}\overrightarrow{U_2}, \quad (6)$$

where $\Delta$ is Laplacian, and $\rho_s, \rho_1, \rho_2$ are the densities of the solid, nonwetting, and wetting phases, respectively.

A digression is now in order. The force component $\partial\tau_{ij}/\partial x_j$ clearly accelerates the entire volume element, with the acceleration $\ddot{\vec{u}}_{el}$ applied to the center-of-mass and appearing on the left-hand side of equation 6. The aggregate stress tensor, defined in equation 5, is then used. This follows Biot (1955, equation 2.10) but departs from Biot (1956a, equation 3.21). In the latter approach, the author writes an individual equation of motion for each phase, separating the forces acting on each. This separation is not strictly required and is, in effect, a choice; for example, as a consequence, it forces one to introduce the additional "coupling" coefficients to keep the equations linked. In Biot's (1956a, equations 3.21 and 4.2) treatment, these coefficients are $\rho_{11}, \rho_{12}, \rho_{22}$, whose practical meaning remained rather speculative even in subsequent works (Molotkov,

2002, p. 752; I. Beresnev, submitted for publication, 2013). On the other hand, the former approach, implemented in equation 6, avoids introducing additional constants and working toward their interpretation, and it keeps the porosity as a natural part of the equation of motion. As discussed in the next section, this approach is equivalent to the "low-frequency" approximation.

The methodological consequence of separating the equations of motion for each phase does not end at this point. The acceleration $\ddot{\vec{u}}_{el}$ of the center-of-mass in the "one-equation" approach, as in equation 6, does not state anything about the individual accelerations of the constituents. Equation 6, therefore, is particularly useful in tracking the motion of phases as one, which is the behavior attributed to the wave of the first kind. On the other hand, the approach of separating the aggregate motion into those of individual phases results in the coupled equations of motion, as many as there are unknown displacements in the phases. This mathematically allows independent solutions for the displacement in each phase, appearing as the propagation of separate waves of the first and second kind (in Biot's case) or additionally the wave of the third kind (in the three-phase system). In a sense, these waves arise from one of the alternative mathematical formulations of the problem and, as also further discussed in the next section, can be thought of as possible in the high-frequency range. Notice that a no-less-rigorous formalism of the propagation of the conventional compressional wave (the wave of the first kind) can thus be achieved at low frequencies, without deducing the waves of the second and third kind and introducing the additional constants. The latter aspect is particularly appealing.

There are reports of the observation of the wave of the second kind (Plona, 1980; Johnson et al., 1982; Gurevich et al., 1999). The wave of the third kind has apparently never been observed. The properties of the P2 and P3 waves appear to be of rather limited practical interest. It is sometimes hypothesized that their presence is necessary to correctly describe the mechanism of seismic attenuation (Mavko et al., 2009, p. 269).

## Compressional-wave propagation velocity

*General equation for velocity*

In thus setting the proper stage methodologically, we can continue with the analysis of equation 6. We now apply the divergence operator to equation 6. Using the identity $\text{div grad } x = \Delta x$, where $x$ is a scalar quantity, this transforms equation 6 to

$$[(1-f)\rho_s + f(S_1\rho_1 + S_2\rho_2)]\frac{\partial^2}{\partial t^2}\text{div}\vec{u}_{el}$$
$$= \Delta\{[(1-f)(\lambda+2\mu) + f(S_1Q_1 + S_2Q_2)]\text{div}\vec{u}$$
$$+ [(1-f)Q_1 + f(S_1R_2 + S_2R_1)]\text{div}\overrightarrow{U_1}$$
$$+ [(1-f)Q_2 + f(S_1R_1 + S_2R_3)]\text{div}\overrightarrow{U_2}\}. \quad (7)$$

Because the divergence of the displacement vector equals the relative volume change (Landau and Lifshitz, 1959, equation 1.6), we now seek the solution of equation 7 in the form of a compressional plane wave of the first kind, in which all phases move as a whole:

$$\text{div}\vec{u}_{el} = \text{div}\vec{u} = \text{div}\overrightarrow{U_1} = \text{div}\overrightarrow{U_2} = C\exp[i(lx + \omega t)].$$
(8)

This is always the case in the low-frequency approximation, in which the velocity dispersion disappears (see Biot, 1956a, Figure 3; Murphy et al., 1993, p. 230). Our theory, therefore, describes the propagation of low-frequency compressional waves with wavelengths much greater than the typical pore diameters. This approximation assumes that at any given moment, the fluid pressure between the adjacent pores equilibrates and there is no flow between them (no "squirt" flow).

The condition (equation 8) for the relative volume changes in the solid and fluid phases can be arrived at purely conceptually. Consider a wave carrying pure volume deformation in a medium consisting of a solid matrix surrounded by uniformly distributed fluid-filled voids, and the wavelength is much greater than the linear dimensions of grains and voids so that all phases are under the same pressure. There can only be three possibilities: the relative volume change in the matrix greater than, smaller than, or equal to that in the fluids. The first possibility would mean that the bulk modulus of the solid is lower than that of the fluids, which, by the nature of geologic solids, is ruled out. Because the fluids are under the same pressure, the second scenario would imply the appearance of empty voids in the medium, which is unphysical. The equality of the relative volume changes (equation 8) thus remains the only possibility. Condition 8 is the self-similarity condition of Gassmann (1951, article 45), postulated by him for pure volume strain.

The self-similarity breaks down if the wavelength becomes comparable to the dimensions of grains and voids, in which case each of them undergoes its own independent volume strain. In that case, independent solutions for the deformations in the solid and liquid phases become possible, mathematically allowing the existence of the waves of the first, second, and third kind. This difference between the characters of the low- and high-frequency solutions for dilatational waves may explain why the waves of the second kind have only been practically observed in the megahertz frequency range, as reported by Gurevich et al. (1999). Mathematically, the difference is demonstrated by the fact that, as frequency tends to zero, the wave of the second kind becomes a diffusion process with its velocity tending to zero while the velocity of the conventional wave has a finite limit (see Biot, 1956a, Figures 3 and 5).

The equality of the amplitudes of the volume strain (the coefficients $C$) in the solid and fluid phases can also be proved formally. The proof is provided in Appendix A.

Substituting equation 8 into equation 7 and completing the algebra, we arrive at the algebraic equation, from which the wave-propagation velocity $\omega/l$ can be retrieved:

$$\frac{\omega^2}{l^2}[(1-f)\rho_s + f(S_1\rho_1 + S_2\rho_2)]$$
$$= (1-f)(\lambda + 2\mu + Q_1 + Q_2)$$
$$+ f[S_1(Q_1 + R_1 + R_2) + S_2(Q_2 + R_1 + R_3)]. \quad (9)$$

Note that the meaning of the coefficients $Q_1, Q_2, R_1, R_2, R_3$ still remains unclear, nor has the capillary-pressure effect been incorporated yet.

One now observes that the velocity determined by equation 9 has to converge, as limiting cases, to the correct velocity in the solid in the case of zero porosity, and the velocity in the fluid for the poros-

ity of one and the full saturation of the respective fluid. Setting $f = 0$ (the case of pure solid) transforms equation 9 to

$$\frac{\omega^2}{l^2}\rho_s = (\lambda + 2\mu + Q_1 + Q_2). \quad (10)$$

To obtain the correct compressional-wave velocity in the solid, we thus have to set

$$Q_1 = -Q_2 \equiv Q. \quad (11)$$

Setting $f = 1$ (the case of pure fluids) transforms equation 9, with equation 11 in mind, to

$$\frac{\omega^2}{l^2}(S_1\rho_1 + S_2\rho_2) = S_1(Q + R_1 + R_2)$$
$$+ S_2(-Q + R_1 + R_3). \quad (12)$$

Here, there are two limiting cases: $S_1 = 1, S_2 = 0$ and $S_1 = 0, S_2 = 1$. The first transforms equation 12 to

$$\left(\frac{\omega^2}{l^2}\right)\rho_1 = Q + R_1 + R_2. \quad (13)$$

The second transforms it to

$$\left(\frac{\omega^2}{l^2}\right)\rho_2 = -Q + R_1 + R_3. \quad (14)$$

Because the compressional-wave velocities in either fluid are independent of each other, we have to set $Q + R_1 = 0$ and $-Q + R_1 = 0$. Adding the two leads to

$$R_1 = 0 \quad (15)$$

and subtracting them to $Q = 0$, the latter, in view of equation 11, meaning that

$$Q_1 = Q_2 = 0. \quad (16)$$

With these constraints, equations 13 and 14 reduce to the correct compressional-wave velocities in the nonwetting and wetting fluids, respectively, and the coefficients $R_2$ and $R_3$ acquire the meaning of the bulk moduli of the nonwetting and wetting fluids, respectively. Equation 4 then expresses Hooke's law for the compression of the fluids.

In Brutsaert's (1964) notation, our coefficient $R_1$ is Brutsaert's $c_{gl}$. The latter author hypothesized the condition $c_{gl} = 0$, without giving any physical or mathematical justification (Brutsaert, 1964, p. 249). We have shown that this condition is required to ensure the correct asymptotics of the compressional-wave velocity in the limiting cases of porosity.

With equations 15 and 16 in mind, equation 9 for the wave velocity in the porous material simplifies to

$$\frac{\omega^2}{l^2}[(1-f)\rho_s + f(S_1\rho_1 + S_2\rho_2)]$$
$$= (1-f)(\lambda + 2\mu) + f(S_1 R_2 + S_2 R_3). \qquad (17)$$

As already noted, the interpretation given to the elastic coefficients appearing in the stress-strain relations of equations 3 and 4 differs from the interpretation of the constants in the respective relations of Biot (1955, equation 4.3; reproduced by Biot, 1956a, equation 2.12), provided by Biot and Willis (1957, equation 21). Our interpretation makes the wave velocity in equation 17 a function of the generic elastic parameters of the constituents only and does not invoke any additional bulk parameters. The porosity directly enters the equation of motion 6 and the velocity equation 17 as an independent variable and is not part of any other coefficients. Conversely, Biot and Willis' constants (their equations 21) in the same stress-strain relations are functions of porosity and require the measurements of bulk moduli in idealized experiments (the "jacketed" and "unjacketed" tests). The level of complexity in acquiring the values from these tests is significantly higher.

*Implementation of capillary pressure*

The velocity equation 17 still treats the two fluid phases as independent. We now assume that the nonwetting and wetting fluids coexist in the pores and are separated by a meniscus. The fluid stresses $\sigma_1$ and $\sigma_2$, therefore, are not independent and are related by Laplace's law of capillary pressure, $p_1 - p_2 = \gamma k$, where $p_1, p_2$ are the fluid pressures, $\gamma$ is interfacial tension, and $k$ is the meniscus' mean curvature. Considering that $\sigma_1 = -p_1$ and $\sigma_2 = -p_2$, we obtain $\sigma_2 = \sigma_1 + \gamma k$. From equations 4, recalling the values of the coefficients found, $\sigma_1 = R_2\varepsilon_1$ and $\sigma_2 = R_3\varepsilon_2$, which leads to

$$R_3\varepsilon_2 = R_2\varepsilon_1 + \gamma k. \qquad (18)$$

The dilatations $\varepsilon_1$ and $\varepsilon_2$ have been assigned the form of a wave carrying a relative volume change in equation 8. Under the effect of this wave, the shape of the menisci undergo a similar bending motion of the form $k_0 \exp[i(lx + \omega t)]$, where $k_0$ is the curvature amplitude. Substituting this expression and equation 8 into equation 18 yields

$$R_3 = R_2 + \frac{\gamma k_0}{C} \quad \text{or} \quad R_2 = R_3 - \frac{\gamma k_0}{C}. \qquad (19)$$

In equation 19, all the quantities are independently fixed except the amplitude $k_0$ of the oscillations of the meniscus curvature. What we have found therefore is that the effect of the incorporation of Laplace's law is to merely constrain the quantity $k_0$ to adjust itself to the propagating volume change to satisfy the condition imposed by equation 19. The latter can then be considered the defining equations for the amplitude of the curvature. Other than constraining the curvature, no other effect on the wave propagation is seen from the relationship between the fluid pressures due to the basic law of capillarity. We then conclude that the presence of capillary pressure does not affect the properties of the wave itself.

Solving equation 17 for $\omega^2/l^2$ results in the compressional-wave propagation velocity in the three-phase system as

$$\frac{\omega^2}{l^2} = \frac{(1-f)(\lambda + 2\mu) + f(S_1 R_2 + S_2 R_3)}{[(1-f)\rho_s + f(S_1\rho_1 + S_2\rho_2)]}. \qquad (20)$$

The velocity values in equation 20 are real: There is not an attenuation mechanism built into the model.

*Modification of velocity equation*

Equation 20 expresses the wave velocity through the elastic constants of the constituents only. What we have obtained is a simple version of the "effective-medium" (Choy, 1999) velocity, in which the moduli governing the propagation of a compressional wave in pure solid ($\lambda + 2\mu$) and pure fluids ($R_2$ and $R_3$) are weighted by the respective volume fractions. Such a result could be arrived at even intuitively; however, the equation was rigorously derived from the first principles of elasticity. The next question is whether it can be made compatible with the body of empirical data.

It is seen that equation 20 needs further modifications. First, one can observe that, in the dry limit of the fluid moduli and densities $R_2, R_3, \rho_1, \rho_2$ tending to zero, the velocity reduces to the compressional-wave speed in pure solid. This is the direct consequence of the way the original energy expansion in equation 1 was written, which made no distinction between the compression of the solid and that of the fluids. The difficulty can be avoided on the following grounds. In equation 20, the numerator can be thought of as the weighting of the solid and fluid moduli by the effect of porosity. In real porous media, possessing extreme geometric complexity at the pore level, such a weighting can be expected to depart from the idealized first-power behavior. We hypothesize that a real porous medium may be described more accurately if we postulate nonlinear weighting as the generalized power of the volume fraction of the phases:

$$\frac{\omega^2}{l^2} = \frac{(1-f)^n(\lambda + 2\mu) + f^m(S_1 R_2 + S_2 R_3)}{[(1-f)\rho_s + f(S_1\rho_1 + S_2\rho_2)]}. \qquad (21)$$

The powers $n$ and $m$ do not have to be the same: One only has to ensure that the proper limiting cases are satisfied; that is, the medium correctly reduces to the pure solid and pure fluids in cases of the porosity of zero and one, respectively. This is still true. The correctness of the hypothesis and the values of the powers can only be ascertained from comparison with empirical data.

Second, even a small presence of a highly compressible nonwetting phase (e.g., gas) significantly lowers the effective bulk modulus of the porous material until the saturation $S_1$ reaches zero. This behavior is not captured by the additivity of the fluid moduli in equation 21, which, again, has its origin in the form of the energy-expansion equation 1 that treats the solid and fluids equally, but it is correctly rendered by using the additivity of the fluid compressibilities (compressibility = inverse bulk modulus) (e.g., Murphy, 1984, equation 4). We are thus constrained to use equation 21 in the equivalent form

$$\frac{\omega^2}{l^2} = \frac{(1-f)^n\left(K + \frac{4\mu}{3}\right) + f^m\frac{1}{\frac{S_1}{R_2} + \frac{S_2}{R_3}}}{[(1-f)\rho_s + f(S_1\rho_1 + S_2\rho_2)]}, \qquad (22)$$

where the identity $\lambda + 2\mu = K + \frac{4\mu}{3}$ for the solid has been used, where $K$ is the bulk modulus of the solid. Equations 21 and 22

are equivalent because, in case of one fluid ($R_2 = R_3$), they are identical. Equation 22 is the one that will be compared to empirical data.

## DETERMINATION OF CONSTANTS AND COMPARISON WITH EMPIRICAL DATA

### Moduli of dry solid frame

Equation 22 predicts the effect of porosity on the reduction of the elastic moduli of pure solid and fluid phases as they combine in a porous aggregate. For example, for the dry material (both fluid phases are infinitely compliant, $R_2 = R_3 = 0$), the effective bulk and shear moduli of the solid skeleton follow as

$$K_{\text{dry}} = (1 - f)^n K \quad \text{and} \quad \mu_{\text{dry}} = (1 - f)^n \mu. \quad (23)$$

An empirical equation for $K_{\text{dry}}$ and $\mu_{\text{dry}}$, irrespective of lithology, was developed by Krief et al. (1990) and has been used for velocity-porosity model calibrations (e.g., Goldberg and Gurevich, 1998):

$$K_{\text{dry}} = (1 - f)^{\frac{3}{1-f}} K \quad \text{and} \quad \mu_{\text{dry}} = (1 - f)^{\frac{3}{1-f}} \mu. \quad (24)$$

Another empirical relation was found by Murphy et al. (1993) for quartz sands and sandstones:

$$K_{\text{dry}} = \begin{cases} 38.18(1 - 3.39f + 1.95f^2), & f \leq 0.35 \\ \exp(-62.60f + 22.58), & f > 0.35 \end{cases}$$
$$\mu_{\text{dry}} = \begin{cases} 42.65(1 - 3.48f + 2.19f^2), & f \leq 0.35 \\ \exp(-62.69f + 22.73), & f > 0.35 \end{cases}, \quad (25)$$

where the moduli are expressed in GPa. The generic relation of Krief et al. (1990) can be used to set the value of the power $n$ in equations 23. Note that Krief et al. (1990)'s relation formally has no meaning at $f = 1$, while our theory is free of this inconvenience.

Figure 1 plots the graphs of $K_{\text{dry}}$ and $\mu_{\text{dry}}$, as functions of porosity, calculated from the theoretical equation 23 with $n = 4$, along with those of Krief et al. (1990) (our equation 24) and Murphy et al. (1993) (our equation 25). The generic values of the moduli for quartz $K = 37$ GPa and $\mu = 45$ GPa (Zinszner and Pellerin, 2007, Table 1-3.2) were used for the calculations in equations 23 and 24. These values will always be used in the following unless otherwise noted. Figure 1 shows that equations 23 with $n = 4$ follow Krief et al. (1990)'s empirical relations almost exactly and are also close to Murphy et al. (1993)'s empirical functions. We conclude that the theoretical dry moduli are well representative of the observations if $n$ is set to four. This value of $n$ will be fixed for all subsequent calculations. For aesthetic reasons, we have not tried noninteger values of $n$.

### Bulk modulus of solid frame saturated with one fluid

The bulk modulus of the solid frame saturated with one fluid (e.g., $R_2 \equiv R, S_1 = 1$) follows from equations 22 as

$$K_{\text{sat}} = (1 - f)^4 K + f^m R. \quad (26)$$

The value of $K_{\text{sat}}$, as a function of porosity and the bulk moduli $K$ and $R$ of the constituents, is typically obtained in practice from Gassmann's formula (e.g., Sheriff and Geldart, 1995, equation 5.11):

$$K_{\text{sat}} = K_{\text{dry}} + \frac{1 - \left(\frac{K_{\text{dry}}}{K}\right)^2}{f\left(\frac{1}{R} - \frac{1}{K}\right) + \frac{1}{K}\left(1 - \frac{K_{\text{dry}}}{K}\right)}. \quad (27)$$

However, this model is not self-contained and has to be used in a two-step manner, in that it requires independent knowledge of $K_{\text{dry}}$. The latter, for example, can be obtained from Krief et al. (1990)'s empirical rule (Goldberg and Gurevich, 1998).

Figure 2 compares the theoretical prediction for $K_{\text{sat}}$ (equation 26), as a function of porosity, with the values of $K_{\text{sat}}$ following from Gassmann's model, in which $K_{\text{dry}}$ has been prescribed according to the rule of Krief et al. (1990). The value of $m$ in equation 26 has been set to zero, and a typical bulk modulus of brine $R = 3$ GPa has been assumed. The two curves in Figure 2 are almost indistinguishable.

With $m = 0$, there is a minor mathematical inconvenience in equation 26 in that the case of $f = 0$ leads to an indeterminateness of $0^0$. This is of little practical concern, though because the power $m$ can be set to an arbitrary small number instead, which avoids the uncertainty altogether by correctly removing the second term in equation 26 for $f = 0$.
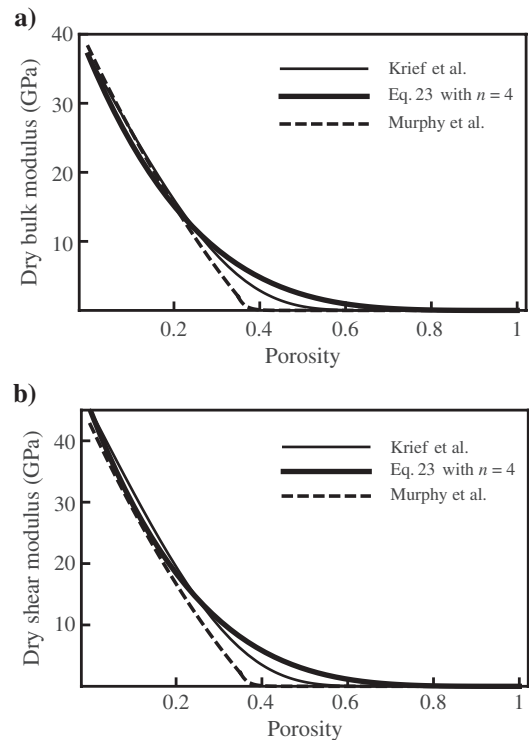


Figure 1. (a) Bulk and (b) shear moduli of the dry solid frame as a function of porosity. Empirical relations of Krief et al. (1990) and Murphy et al. (1993) are compared with theoretical equation 23 with $n = 4$.

One now can rewrite equation 22 as

$$\frac{\omega^2}{l^2} = \frac{(1-f)^4\left(K + \frac{4\mu}{3}\right) + \frac{1}{\frac{S_1}{R_2} + \frac{S_2}{R_3}}}{[(1-f)\rho_s + f(S_1\rho_1 + S_2\rho_2)]},$$
(28)

which requires no further modification. Again, albeit it appears that, at $f = 0$, the fluid moduli $R_2$ and $R_3$ contribute to the wave velocity, this is of no concern because the power $m$ can be thought of as being a small number instead of zero.

## Compressional-wave velocity in porous material saturated with one fluid

In the case of one fluid (e.g., $R_2 \equiv R$, $S_1 \equiv 1$), equation 28 reduces to

$$\frac{\omega^2}{l^2} = \frac{(1-f)^4\left(K + \frac{4\mu}{3}\right) + R}{(1-f)\rho_s + f\rho_1}.$$
(29)

As a benchmark, we can again use the conventional two-step compressional velocity calculation based on Gassmann's relation:

$$V_P^2 = \frac{K_{sat} + \frac{4\mu_{dry}}{3}}{(1-f)\rho_s + f\rho_1},$$
(30)

where $K_{sat}$ is the Gassmann value (Sheriff and Geldart, 1995, p. 112). An assumption is made that the presence of pore-filling fluid does not alter the shear modulus of the frame; that is, $\mu_{sat} = \mu_{dry}$ (Goldberg and Gurevich, 1998). The calculation of modulus $K_{sat}$ in Gassmann's equation 27 again uses the value of $K_{dry}$ obtained from Krief et al. (1990).

Another empirical relation is the Raymer-Hunt-Gardner rule, known to fit the measured compressional-wave velocities $V_P$ in water-saturated sandstones very closely, at least in the porosity range of 0 to 0.4 (Mavko et al., 2009, p. 353 and Figure 7.4.2),

$$V_P = \begin{cases} (1-f)^2 V + f V_1, & f < 0.37 \\ \frac{1}{[(1-f)\rho_s + f\rho_1]V_P^2} = \frac{f}{\rho_1 V_1^2} + \frac{(1-f)}{\rho_s V^2}, & f > 0.47 \end{cases}$$
(31)

where $V = [(K + 4\mu/3)/\rho_s]^{1/2}$ and $V_1 = (R/\rho_1)^{1/2}$ are the wave-propagation speeds in the pure solid and pure fluid, respectively.

Figure 3 compares the theoretically predicted velocities (equation 29), as a function of porosity, with those calculated from the Gassmann-Krief rule (equation 30) and the Raymer-Hunt-Gardner rule (equation 31). The same moduli for quartz and brine were used as before, and the densities of quartz $\rho_s = 2650$ kg/m$^3$ (Zinszner and Pellerin, 2007, Table 1-3.2) and brine $\rho_1 = 1100$ kg/m$^3$ were taken. The theory agrees well with the empirically based relations.

## Compressional-wave velocity in porous material saturated with two fluids

The full equation 28 can be checked against experimental data obtained for the case of the solid saturated with two fluids. Cadoret et al. (1995) measure the wave velocities in limestone saturated with air and water with a varying degree of water saturation. Figure 4 compares their experimental data (Cadoret et al., 1995, Figures 3 and 4) with the predictions from equation 28, as a function of water saturation $S_2$. The open and filled circles correspond to the two sets of experiments that used different methods of achieving the given water saturation.

What was measured in the experiments were the velocities $V_E$ of extensional waves and $V_S$ of shear waves in a resonant bar; we converted these velocities to $V_P$ using the formula provided by the authors (Cadoret et al., 1995, equation 1). The rock sample was characterized as a bioclastic limestone composed of calcite with the porosity of $0.30 \pm 0.02$ (Cadoret et al., 1995, Table 1 and p. 9796). In the theoretical calculations, we therefore assumed the generic values of the elastic coefficients for calcite: $K = 70$ GPa, $\mu = 30$ GPa, and $\rho_s = 2710$ kg/m$^3$ (Zinszner and Pellerin, 2007, Table 1-3.2). Because the samples were saturated with water and air, the following constants were assumed for the fluids: $R_2 = 0.1$ MPa, $R_3 = 2.25$ GPa, $\rho_1 = 1.2$ kg/m$^3$, and $\rho_2 = 997$ kg/m$^3$. The porosity was taken as 0.30.
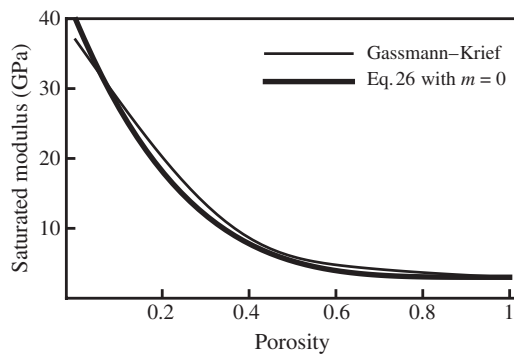
Figure 2. Bulk modulus of the solid frame, saturated with one fluid, as a function of porosity. Theoretical equation 26 with $m = 0$ is compared with the Gassmann-Krief combined equation 27.
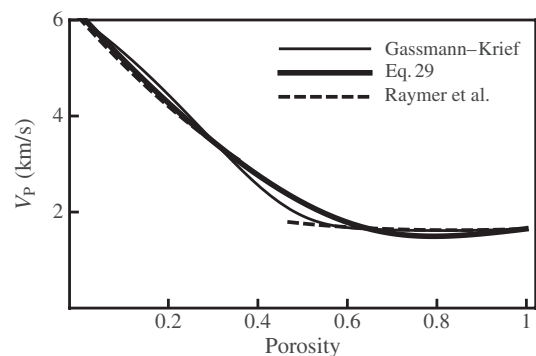
Figure 3. Compressional-wave velocity in the porous material, saturated with one fluid, as a function of porosity. Theoretical equation 29 is compared with the Gassmann-Krief and Raymer-Hunt-Gardner rules.

Figure 4a compares the data with the theoretical values of wave velocity calculated for the generic limestone constants. Although the shapes of the theoretical and experimental dependences match well, there is nearly a constant magnitude shift. Two hypotheses can be put forward in a way of a possible explanation of the latter. First, because no specific values of $K$ and $\mu$ for the mineral grains were provided, we have assumed the generic values for pure calcite. Any departure of these from the actual values for the grains in the sample will cause a proportional shift in the wave speed. As an illustration, Figure 4b shows the theoretical curve with the values of $K$ and $\mu$ both reduced to 0.65 of their generic values, as well as with the porosity increased to 0.32 (still within the measurement uncertainty). This makes the match between the theory and experiment nearly perfect and demonstrates the effect of the solid moduli and porosity on the velocity.

Second, the experiment measured the speeds of the extensional and shear waves in a bar of rock, from which the compressional-wave speeds were calculated. The ratio $V_P/V_E$ tends to infinity as Poisson's ratio of the rock approaches 0.5 (e.g., Mavko et al., 2009, p. 81). Cadoret et al. (1995, p. 9792) are cognizant of the potential conversion error that this can induce, and they limit their data to samples with Poisson's ratio lower than 0.4. The amount of a possible residual systematic error is nonetheless not quantified and may contribute to the magnitude difference seen in Figure 4a.

It is worth noting in this regard that compilations of experimental data put the P-wave velocities for typical limestones in the range between 3.5 and 6 km/s (Sheriff and Geldart, 1995,

Figure 5.5). Evidently, the experimental values in Figure 4 are below this range.

## CONCLUSIONS

The three questions posed in the Introduction have been answered as follows. An effective-medium-type theory has been deduced from principles of linear elasticity to describe the propagation of the waves of the first kind in a porous solid filled with a nonwetting and wetting fluids. The theory has been made to explicitly comply with Laplace's law of capillary pressure at the pore level and resulted in the determination of the compressional-wave velocity. The model contains only the elastic constants of the constituent solid and fluid phases and does not need to introduce any bulk constants defined for the solid frame or poorly understood "mass-coupling" coefficients. This provides a high degree of convenience in its practical use.

The theory shows no effect of the interfacial tension between the fluids on the seismic velocity.

The theory is successfully reconciled with empirical data if two modifications are introduced, both justified on the physical grounds. First, the effect of porosity on the reduction of moduli of the solid and fluid phases in the aggregate is hypothesized to take the nonlinear form of a power law, with separate powers governing the weighting of the solid and fluid moduli by the porosity. Second, the fact was used that the additivity of the compressibilities, rather than the additivity of the bulk moduli, describes the effective bulk modulus of the fluids. The incorporation of these two simple principles has led to an equation for the compressional-wave velocity that can be compared with empirical data.

The general form of the resulting velocity equation 22 still contains two undetermined constants (the powers $n$ and $m$) that are unresolvable from within the theory. These constants have to be calibrated against observations. For example, equation 28, with $n = 4$ and $m = 0$, without any more tunable parameters and only using the generic elastic constants of the constituent phases, seems to be widely applicable in its ability to closely match a variety of common empirical relations, observed for at least well-cemented rocks such as sandstones or limestones. Specifically, it is able to reproduce the bulk modulus of the dry frame and the bulk modulus of the frame saturated with one fluid, both in the full range of porosity. It successfully describes the observed compressional-wave velocity in rocks saturated with one fluid, in the full range of porosity. It also compares favorably with the observed compressional-wave velocity in rocks saturated with air and water, in the full range of water saturations.

It is possible that calibrating the internal parameters $n$ and $m$ in the general model to different rock types can make it applicable to other classes of rock as well. The lack of one universal velocity equation valid for all types of rock is indicative of the limitations of the direct extension of the classic model of the linear elastic solid, formulated in equation 1, in its ability to describe the variety of properties of realistic porous rocks solely through the elastic moduli of the constituent phases. Our analysis has nonetheless shown that the model can possibly be tuned to a desired class of rock. It appears that, by the complexity of the mechanical constitutive behaviors of the geologic materials, a theory of wave propagation in geologic media only using the elastic moduli of the pure phases inevitably acquires a semi-empirical character, in the sense that its internal
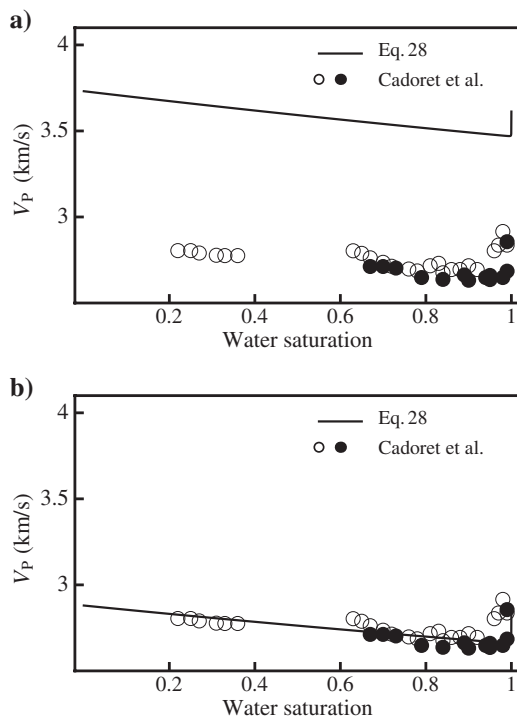


Figure 4. Compressional-wave velocity in a limestone saturated with air and water, as a function of water saturation. Theoretical equation 28 is compared with data from Cadoret et al. (1995). (a) Generic values of calcite moduli and $f = 0.30$ used in the theory. (b) Calcite moduli reduced by 35% and $f = 0.32$.

parameters cannot be deduced from the theory itself but need to be calibrated against the reality.

## APPENDIX A

## PROOF OF CONDITION 8

The relation between the amplitudes $C_1$ and $C_2$ of the volume strain in the solid, $\mathrm{div}\vec{u} = C_1 \exp[i(lx + \omega t)]$, and the fluid, $\mathrm{div}\vec{U} = C_2 \exp[i(lx + \omega t)]$, can be established from the original Biot theory (see Biot, 1956b, equation 6.2). Substituting into the first of the equations of motion 6.1 of Biot (1956b) and keeping in mind the notation $z \equiv l^2 V_c^2/\omega^2$ (Biot, 1956b, equation 6.3), where $V_c$ is Biot's reference velocity (Biot, 1956a, equation 5.4), one obtains

$$C_1 = C_2 \frac{Q \frac{z}{V_c^2} - \rho_{12} - \frac{ibF}{\omega}}{-P \frac{z}{V_c^2} + \rho_{11} - \frac{ibF}{\omega}}, \tag{A-1}$$

where $Q$ and $P$ are Biot's elastic coefficients, $F$ is the complex frequency-dependent function describing the deviation of the friction force, exerted by the fluid on the pore wall, from the Poiseuillian value, and $b$ is the friction coefficient. The values of $z$ are obtained as the roots of the quadratic dispersion equation (Biot, 1956b, equation 6.3), one for the wave of the first kind and one for the wave of the second kind. The explicit roots for the complex-valued compressional-wave velocities $V_P^2 = \omega^2/l^2$ obtained by Beresnev (I. Beresnev, submitted for publication, 2013) as $V_P^2 = a_2^* + ib_2^*$, where $a_2^*$ and $b_2^*$ are frequency-dependent coefficients (I. Beresnev, submitted for publication, 2013, equations 12–15; the original notation is kept). By recalling the definition of Biot's characteristic frequency $\omega_c \equiv b/(f\rho_f)$, where $\rho_f$ is the density of the fluid (Biot, 1956a, equation 7.4), the low-frequency range is introduced as $\omega \ll \omega_c$. In this limit, the equation for $V_P^2$ for the wave of the first kind, $V_P^{(1)2}$, reduces to

$$V_P^{(1)2} = V_c^2(1 - iF_i/2), \tag{A-2}$$

where $F_i \equiv \mathrm{Im}F$ (I. Beresnev, submitted for publication, 2013). Replacing $\frac{z}{V_c^2} = \frac{1}{V_P^2}$ in equation A-1 using $V_P^{(1)2}$ from equation A-2 leads to

$$C_1^{(1)} = C_2^{(1)} \frac{V_c^2 f\rho_f \frac{\omega_c}{\omega} + [P + Q - V_c^2(\rho_{11} + \rho_{22})]}{V_c^2 f\rho_f \frac{\omega_c}{\omega}}, \tag{A-3}$$

where the superscripts in $C_1^{(1)}$ and $C_2^{(1)}$ indicate that these are amplitudes of the volume strain in the solid and the fluid in the wave of the first kind. In transitioning from A-1 to A-3, the equality $\frac{b}{\omega} = f\rho_f \frac{\omega_c}{\omega}$ was used and the conditions for the low-frequency approximation were observed: $\omega \ll \omega_c$; $F_i \to 0$, $F_r \equiv \mathrm{Re}F \to 1$, and $\frac{\omega_c}{\omega} F_i = O(1)$, all as $\omega/\omega_c \to 0$ (Biot, 1956b, Figures 2 and 4; I. Beresnev, submitted for publication, 2013). The real part of the multiplier on the right-hand side of equation A-3 is equal to

one, and the imaginary part tends to zero as $\omega/\omega_c \to 0$. The multiplier therefore is equal to the real-valued one, which shows that the amplitudes $C_1^{(1)}$ and $C_2^{(1)}$ are identical.

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