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The path-independent J-integral and the crack-extension energy-release rate

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ABSTRACT

The equilibrium, path-independent J-integral is commonly interpreted as the rate of decrease in the potential energy of an elastic body, containing a crack, per unit area of the new surface created by crack extension. Such an interpretation encounters difficulties. The correct, contradiction-free description of the energy-release rate is the approach based on the dynamic energy flow into the crack tip. As a corollary, it leads to the correct quasistatic criteria for brittle-fracture propagation.

1. Introduction

The J-integral in its commonly used form was introduced into the two-dimensional crack mechanics by Rice (1968a). If Γ is an arbitrary contour around the tip of a notch or a crack, beginning and ending on their opposite sides, J is defined as

$$J = \int_{\Gamma} \left(w dy - T_i \frac{\partial u_i}{\partial x} d\Gamma \right) \tag{1}$$

where w is the elastic-strain energy per unit volume, u_i are the components of the displacement vector, $T_i = \sigma_{ij} n_j$ are the components of the traction vector, σ_{ij} is the stress tensor, and n_j are the components of the unit normal external to Γ (Fig. 1). Assuming the equilibrium condition $\frac{\partial \sigma_{ij}}{\partial x_j} = 0$, the independence of the value of the integral (1) on the specific choice of the path Γ is readily proved (Rice, 1968a, pp. 379–380, reproduced in a more detailed form by Anderson, 2005, section A3.2).

It has been common to interpret the J-integral as the rate of decrease in the potential energy of the body per unit area of new rupture surface created by the crack extension, $G \equiv -\partial \Pi / \partial A$, also termed the energy-release rate or crack-extension force. In the two-dimensional case, G reduces to $-\partial \Pi / \partial a$, a being the crack length, and is the energy-release rate per unit width perpendicular to the (x, y)-plane in Fig. 1. However, it can be argued that such an interpretation is less obvious and contradictory. It thus is important to understand how the energy-release rate should be correctly described.

2. J-integral in the absence of crack

Consider an example. An elastic body is subject to the state of deformation in which

$$\frac{\partial u_x}{\partial x} = \frac{\partial u_y}{\partial x} = 0 \tag{2}$$

(for example, simple shear produced by σ_{yx} stresses). Take a closed loop around an arbitrary point A′ in the body, as shown in Fig. 2, where Γ_3 and Γ_4 are parallel to the x-axis. There is no crack or notch in the body: the dashed line AA'B of length a is purely fictitious. The J-integral along any such closed loop is zero (Rice 1968a, equation 3c; Anderson, 2005, equation A3.19). The path-independence of the integral along a particular segment around the "tip" A′, beginning under the line BA′ and ending above AA' (e. g, contours Γ_1 or Γ_2), is proved in the same manner as for the segments around the tip of a real crack or notch, as follows.

The zero value of J along the loop is written as $J = J_1 + J_2 + J_3 + J_4 = 0$, where the subscript refers to the integrals along Γ_1, Γ_2, Γ_3, and Γ_4 in Fig. 2, respectively. Consider the value of the integrand on Γ_3 and Γ_4, on which $dy = 0$ and the normal has the n_y-component only. The integrand reduces to $-\sigma_{ij} n_j \frac{\partial u_i}{\partial x} = -\sigma_{xy} n_y \frac{\partial u_x}{\partial x} - \sigma_{yy} n_y \frac{\partial u_y}{\partial x}$, which is zero because of (2), and so are J_3 and J_4. It follows that $J_1 = -J_2$, and, reverting the path direction on Γ_2 to counterclockwise, $J_1 = J_2$. This proves the path independence of J along any contour around A' in the example considered, and this does not require any presence of crack or notch.

The definitions of J in Figs. 1 and 2 are in no way different except that Fig. 2 does not contain a crack, the line AA'B being fictitious. If the J-

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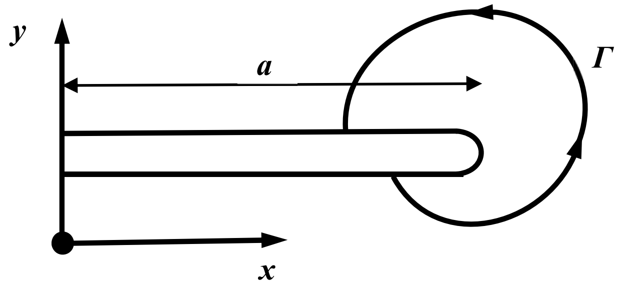

Fig. 1. An arbitrary contour around the tip of a notch or a crack.

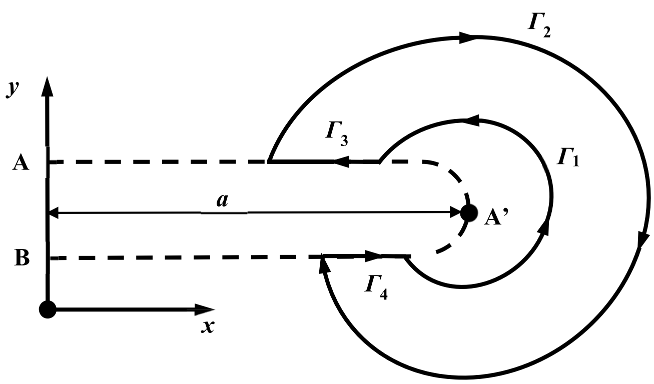

Fig. 2. A closed contour around a point A′ in an elastic body.

integral is identical to $-\partial\Pi/\partial A$, it must in the latter case be zero, because there are neither crack nor its extension present, and there is no energy change. Let us calculate the value of J along the particular contour depicted in Fig. 3, beginning at point 1 and proceeding to point 8. The segments 2–3 and 6–7 are parallel to the x-axis, and segments 1–2, 3–6, and 7–8 to the y-axis. The equality to zero of the entire integrand along 2–3 and 6–7 has just been proved. Along 1–2, 3–6, and 7–8, the normal has the n_x-component only; the second term of the integrand in (1) is $-\sigma_{xx}n_x\frac{\partial u_x}{\partial x}-\sigma_{yx}n_x\frac{\partial u_y}{\partial x}$, which is still zero because of (2). The J-integral along the entire contour reduces to $\int\limits_{\Gamma_{1-2}+\Gamma_{3-6}+\Gamma_{7-8}} wdy$. The integrals along 1–2 and 7–8 will cancel with the ones along 3–4 and 5–6, respectively. The cancellation is owing to the fact that w is positive and is the same in each respective pair due to symmetry, but the direction of integration is opposite. Hence the J-integral is simply $\int\limits_{y_4}^{y_5} wdy$ and is non-zero because w is always positive. We deduce that the path-independent value of the J-integral around the point A′ is non-zero, contradicting the statement that it be zero if the integral represents the energy-release rate. We conclude that the J-integral cannot describe the potential-energy change.

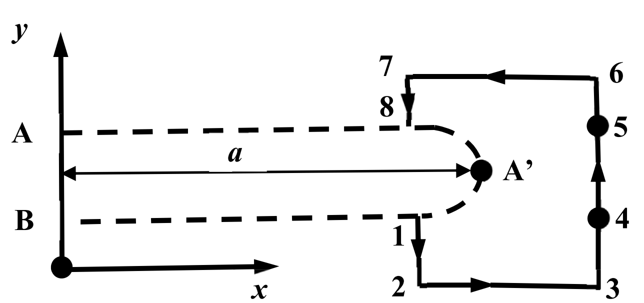

Fig. 3. Specific contour around A′ used in the text.

Note that, by letting points 1 and 8 in Fig. 3 approach each other, we can eventually make the width of the virtual crack infinitely small (point A will coincide with point B). The integral along segments 1–2 and 7–8 in this limit will exactly cancel with the contribution from 3–6. The value of the J-integral will hence become zero. This is to be expected, because, in the limit considered, the loop closes, but the J-integral along any such loop is always zero. The zero result still remains path-independent. In interpreting this fact, one should keep in mind, though, that the concept of the integral, originally developed by Rice and always representing the potential-energy change, equally applies to finite-width notches as in the example considered in Fig. 3 (Rice, 1968a, Fig. 1). As one can see, such an application to a finite notch leads to a paradox.

Rice (1968b, equations 62 and 68) and Anderson (2005, section A3.3) do provide proofs of the equality of the J-integral to the energy-release rate. The root of the apparent paradox may lie in the starting points of their arguments. Both authors begin with the definition of the potential energy of the body as

$$\Pi=\int\limits_V wdV-\int\limits_S T_iu_idS \tag{3}$$

where V is the volume of the body bounded by the surface S (Rice, 1968b, equation 50; Anderson, 2005, equation A3.20) (the convergence of the integrals is assumed). Let us consider the case of linear elasticity. The principle of virtual work states that, in the absence of body forces,

$$\int\limits_V \sigma_{ij}\varepsilon_{ij}dV=\int\limits_S T_iu_idS \tag{4}$$

where ε_{ij} is the strain tensor (Spencer, 1980, equation 7.34 and p. 103). Noticing that $\sigma_{ij}\varepsilon_{ij}=2w$ (Landau and Lifshitz, 1959, equation 4.10), Eq. (4) is re-written as

$$2\int\limits_V wdV=\int\limits_S T_iu_idS \tag{5}$$

which simply is the statement of the fact that, in the absence of body forces, elastic energy of the body can only be acquired through the application of tractions on its boundaries (Love, 1944, Article 120; Love calls the statement the "theorem concerning the potential energy of deformation"). We derived this statement using the principle of virtual work, although it also immediately follows from the theorem. The replacement of the surface integral in (3) with its expression in (5) results in the potential energy that is always necessarily negative. Such a conclusion indicates the flaw in the argument and contradicts the definition of the potential energy "stored up in the body by the strain" by Love (1944, p. 95 and Article 120) as just the first term in Eq. (3). The correctly defined energy is always positive (Love, 1944, pp. 99 and 171).

The necessarily positive nature of the elastic potential energy is further illustrated by the simple example of the deformation of a linear spring. In this case, the energy is $\Pi=(1/2)kd^2$, where k is the spring constant and d is the amount of extension or compression (Young and Freedman, 2008, equation 7.9 and Figure 7.14). The elastic energy (unlike, for example, the gravitational energy: Young and Freedman, 2008, equation 7.2) is never negative, regardless of the sign of deformation.

Unsatisfactory nature of the approach to the potential-energy change based on the J-integral is also illustrated by the fact that it requires the assumption of equilibrium, whereas crack propagation is fundamentally a non-equilibrium phenomenon.

3. Energy flow into the crack tip

What then is the correct description of the energy-release rate? Freund (1972) developed the approach, later reproduced by Aki and

Richards (1980), for the energy flow into the crack tip. Consider a crack propagating with the velocity v in the $+x$-direction, as well as a surface S_t surrounding the tip and traveling with it (Fig. 4). If the energy flow g is defined as the time derivative of the energy inside S_t, then in the immobile coordinate system and in the absence of friction, the quantity g is given by

$$g = -\lim_{S_t\to 0}\int_{S_t}\left(\sigma_{ij}n_j\dot{u}_i + \frac{1}{2}\sigma_{ij}u_{i,j}v_n + \frac{1}{2}\rho\dot{u}_i\dot{u}_i v_n\right)dS \tag{6}$$

(Freund, 1972, equation 13; Aki and Richards, 1980, equation 15.14) (note that the second term in Aki and Richards' contains a typo). Here ρ is the density, v_n is the normal component of velocity of a point on S_t, the dot represents the time derivative, and the comma represents the spatial-coordinate derivative.

Note that the quantity g in (6), unlike the J-integral, always correctly equals zero in case of equilibrium (with or without a crack), meaning the absence of energy change. Indeed, the velocity components $\dot{u}_i$ and v_n then are zero, and so is the integral in (6).

The integral in (6) is then calculated by representing the surface S_t as a rectangular box, as shown in Fig. 5, and switching to the coordinate system $x' = x - vt$ moving with the tip. Because the velocity v_n is zero in the moving coordinates, the second and third terms in (6) vanish. Using the continuity of stress components σ_{ix} and σ_{iy} in the y-direction across the crack plane and the continuity condition $[\sigma_{ix}(\delta,0)\dot{u}_i(\delta,y) - \sigma_{ix}(-\delta,0)\dot{u}_i(-\delta,y)]\to 0$ at $\delta\to 0$, the remaining integral, calculated per unit width perpendicular to the (x', y)-plane, reduces to

$$g = \lim_{\delta\to 0}\int_{-\delta}^{\delta}\sigma_{iy}(x',0,t)\left[\dot{u}_i(x',+0,t) - \dot{u}_i(x',-0,t)\right]dx' \tag{7}$$

(Freund, 1972, equation 16; Aki and Richards, 1980, equation at the bottom of p. 862; note the incorrect sign in Freund's equation). This is the energy flowing per unit time into the crack tip.

Since the crack surface has been assumed traction-free, the integral in (7) should in reality be evaluated from 0 to δ, where the stress concentration ahead of the crack tip takes place.

Upon multiplying (7) by dt, we obtain the energy spent to propagate the fracture a small distance vdt, which is of the same order as δ as dt and δ both tend to zero. Let us denote this energy quantity dA_δ:

$$dA_\delta = \lim_{\delta\to 0}\int_0^{\delta}\sigma_{iy}(x',0,t)[u_i(x',+0,t+dt) - u_i(x',-0,t+dt)]dx' \tag{8}$$

Here we have also made use of the approximation $u_i(x',+0,t+dt) - u_i(x',+0,t) \approx u_i(x',+0,t+dt)$ and the same for $u_i(x',-0,t+dt)$, which is reasonable because most of the displacement takes place after, not prior to, the crack formation. The meaning of Eq. (8) is clear. It represents the work of tractions performed on deforming the medium ahead

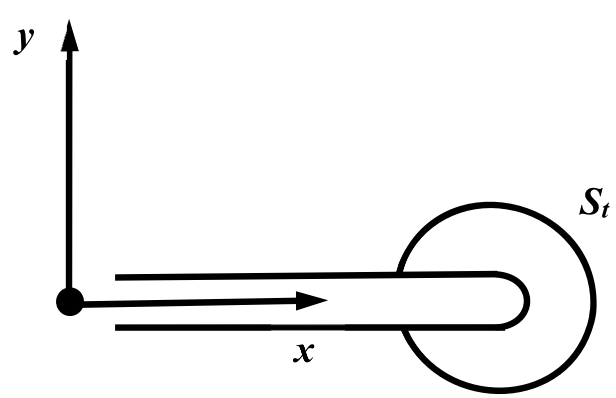

Fig. 4. The tip of a propagating crack surrounded by surface S_t.

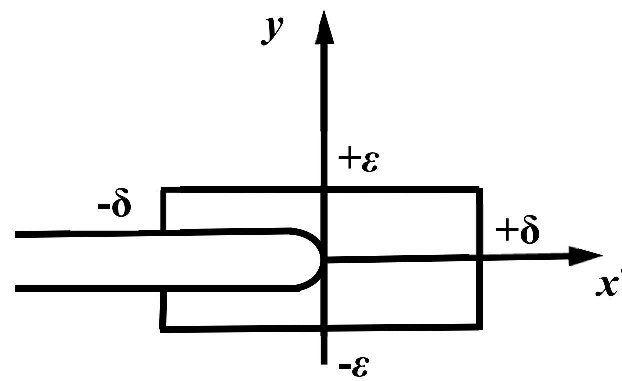

Fig. 5. The surface S_t used to derive Eq. (7).

of the crack tip. In that sense, Eq. (8) can be considered a postulate not requiring any derivation. This work should equal the energy needed to create two new fracture surfaces (two sides of the crack) of length δ.

Note that Eq. (8) is the same as the one written by Sedov (1997, p. 1268, equation 3.16 with the respective change in notation, although its derivation is contradictory), except that there is a factor of ½ in front of the integral in Sedov's expression. Sedov's derivation is sketchy and difficult to follow or verify: the appearance of the factor may be due to a mistake. What is important is that a conceptually correct expression was historically first provided. The 1997 version is the English translation of the Russian original of Sedov's book, whose first edition appeared in 1968. Sedov's result (no reference to its origin was given) thus pre-dates Freund's work.

In the general case of tractions (for example, friction) present at crack surfaces, an integral over the entire crack length a, $\int_{-a}^{0}\sigma_{iy}(x',0)[\dot{u}_i(x',+0) - \dot{u}_i(x',-0)]dx'$, will be added to the right-hand side of Eq. (7).

In unit time, the crack tip travels the distance v. The energy flow g is therefore related to the energy-release rate G as $g = 2Gv$, where the factor of two accounts for the two new surfaces created (Aki and Richards, 1980, equation 15.21).

An example of quasistatic calculation using Eq. (8) is given by Sedov (1997, p. 1270, equation 3.18). Consider an Irwin's Mode-I (extension-opening in the y-direction) brittle crack without friction. Because of the symmetry, the u_x-component of displacement is an even function of y and the u_y-component is an odd one. Eq. (8) becomes

$$dA_\delta = 2\lim_{\delta\to 0}\int_0^{\delta}\sigma_{yy}(x',0,t)u_y(x',+0,t+dt)dx' \tag{9}$$

In the (r, θ) polar coordinate system centered at the crack tip, in which the angle θ is measured counterclockwise from the x-axis in Fig. 5, Irwin's formulae for brittle fracture apply:

$$\begin{aligned}
\sigma_{yy} &= \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right),\\
u_y &= \frac{2K_I(1+\nu)}{E}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left(2-2\nu-2\cos^2\frac{\theta}{2}\right)\quad \text{(plane strain)},\\
u_y &= \frac{K_I(1+\nu)}{E}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left(\frac{4}{1+\nu}-2\cos^2\frac{\theta}{2}\right)\quad \text{(plane stress)},
\end{aligned} \tag{10}$$

where K_I is the Mode-I stress-intensity factor, E is Young's modulus, and ν is Poisson's ratio (Anderson, 2005, Tables 2.1 and 2.2). In utilizing Irwin's equations, valid for a static crack, in Eq. (9), which was derived for the fracture propagating at a constant speed, we make an assumption that such a combination is possible.

In applying Eqs. (10) to (9), the stress is evaluated at the tip ($\theta = 0$) at the time t, that is, before the fracture has opened. However, since the displacement is taken at $t + dt$, that is, when the fracture has already formed, it should be evaluated at $\theta = \pi$. Completing the calculations then leads to

$$dA_\delta = \frac{4K_I^2(1-\nu^2)}{\pi E}\delta \quad \text{(plane strain)}$$

$$dA_\delta = \frac{4K_I^2}{\pi E}\delta \quad \text{(plane stress)} \tag{11}$$

for any small δ.

The work dA_δ equals the energy required to create two new surfaces of length δ. Denoting γ the surface energy of the material per unit area (known as fracture energy), we write

$$dA_\delta = 2\gamma\delta \tag{12}$$

for each equation in (11) (the unit width is still assumed). The stress-intensity factor defined by Eq. (12) is called the critical one K_{Ic}, as this is the value required to extend the crack. On the other hand, the quantity 2γ is the critical energy-release rate G_c (known as fracture toughness), as it defines the energy supply per unit area required to move the crack (Anderson, 2005, equations 1.2, 2.15b, and 2.18 b). Eq. (12) thus relates the critical values of the stress-intensity factor and the energy-release rate. Combining (11) and (12) then leads to

$$G_c = \frac{4}{\pi}\frac{K_{Ic}^2}{E'} \tag{13}$$

where E' equals $E/(1-\nu^2)$ for plane strain and E for plane stress. This is the well known quasistatic relationship between G_c and K_{Ic} (Anderson, 2005, equation 2.56), except that the factor of $4/\pi$ appears in Eq. (13). The nature of this discrepancy is not entirely clear. It may have to do with the fact, stated earlier, that, to arrive at (13), Irwin's expressions (10) for a static crack have been applied to Eq. (9) derived for a crack propagating at a constant speed. The speed can always be assumed to be close to zero, though. The somewhat artificial combination of the static relationship with the concept based on energy flow into the tip may be responsible for the coefficient, albeit insignificantly different from unity ($4/\pi \approx 1.27$). The appearance of this coefficient, whose value is close to one, is of minor concern compared to the well known generally unsatisfactory nature of the linear elastic fracture mechanics, on which the concept of the stress-intensity factor is based, because it leads to stress singularities.

The application of the energy-flow approach, based on Eqs. (7) and (8), thus leads to the correct quasistatic fracture-propagation criterion. The same result (13) (without the $4/\pi$ factor) was obtained from the original concept of g (Eq. (7)), but in a different way, by Freund (1972, equations 26–27), although the author does not provide enough detail of the derivation. It is apparently an unknown fact that Sedov's calculation, also based on the dynamic energy flow into the tip, appeared earlier.

4. Summary

The interpretation of the path-independent J-integral as the rate of change in the potential energy of the body per unit area of crack extension (the energy-release rate) encounters difficulties. The treatment based on the energy flow into the tip of a propagating crack is contradiction-free. When applied to quasistatic brittle fracture, it leads to the correct criterion of crack propagation.

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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