

SYSTEMS OF EQUATIONS

1. SYSTEMS OF EQUATIONS

A system of equations is a finite set of equations containing more than one variable. We sometimes refer to a system of equations as a set of *simultaneous* equations in the sense that we seek values for the variables such that all the equations are true at the same time, i.e. simultaneously. We seek a set of common values for the variables which make all the equations true. Linear systems of equations are systems in which all equations can be expressed in terms of addition and subtraction of multiples of the unknown variables. Nonlinear systems of equations are characterized by having terms which cannot be formed by simply adding together multiples of the unknown variables. For example, possible unknowns might be exponentiated, multiplied together, or be arguments of a nonlinear function.

1.1. Examples of linear equations. Here are some examples of systems of linear equations. Each system has a number of equations and a number (not necessarily the same) of variables for which we would like to solve. The variables are typically represented by letters such as x , y , and z . The first four systems have two equations and two variables. The fifth one has three equations and three variables. The sixth one has two equations and three variables, while the last one has three equations and two variables.

a.
$$\begin{aligned} 2x + y &= 5 \\ -4x + 6y &= -2 \end{aligned}$$

b.
$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 - x_2 &= 1 \end{aligned}$$

c.
$$\begin{aligned} 4x + 3y &= 24 \\ 12x + 9y &= 36 \end{aligned}$$

d.
$$\begin{aligned} x + y + z &= 6 \\ 3x - 2y + 4z &= 9 \\ x - y - z &= 0 \end{aligned}$$

e.
$$\begin{aligned} x_1 + x_2 + x_3 &= 5 \\ x_1 - x_2 &= 2 \end{aligned}$$

f.
$$\begin{aligned} x_1 + x_2 &= 5 \\ 2x_1 - 3x_2 &= -5 \\ 4x_1 - x_2 &= 5 \end{aligned}$$

1.2. Examples of nonlinear equations. Here are some examples of nonlinear equations.

a.
$$\begin{aligned} x + y^2 &= 5 \\ 2x + y &= 4 \end{aligned}$$

b.
$$\begin{aligned} 6x_1^{-1/2}x_2^{1/3} &= 4 \\ 4x_1^{1/2}x_2^{-2/3} &= 3 \end{aligned}$$

2. SOLUTIONS FOR SYSTEMS OF EQUATIONS

2.1. Definition of a solution to a statement of equation. A **solution** of a system of equations consists of values for the variables that reduce each equation of the system to a true statement. To **solve** a system of equations means to find all solutions of the system.

2.2. Examples of solutions.

2.2.1. *Linear example a.* Consider system a in section 1.1 above.

$$\begin{aligned} 2x + y &= 5 \\ -4x + 6y &= -2 \end{aligned}$$

If we substitute (2,1) for (x, y) in the system we obtain:

$$\begin{aligned} 2(2) + (1) &= 5 \\ -4(2) + 6(1) &= -2 \end{aligned}$$

2.2.2. *Linear example b.* Consider system b in section 1.1 above.

$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 - x_2 &= -1 \end{aligned}$$

If we substitute (4/5, 9/5) for (x₁, x₂) in the system we obtain:

$$\begin{aligned} 2\left(\frac{4}{5}\right) + 3\left(\frac{9}{5}\right) &= \frac{8}{5} + \frac{27}{5} = \frac{35}{5} = 7 \\ \frac{4}{5} - \frac{9}{5} &= -\frac{5}{5} = -1 \end{aligned}$$

2.2.3. *Nonlinear example a.* Consider system a in section 1.2 above.

$$\begin{aligned} x + y^2 &= 5 \\ 2x + y &= 4 \end{aligned}$$

If we substitute (1, 2) for (x, y) in the system we obtain

$$\begin{aligned} 1 + 2^2 &= 5 \\ 2(1) + 2 &= 4 \end{aligned}$$

Another solution of the system is given by (x, y) = (11/4, -3/2) as can be seen as follows:

$$\begin{aligned} \frac{11}{4} + \left(\frac{-3}{2}\right)^2 &= \frac{11}{4} + \frac{9}{4} = \frac{20}{4} = 5 \\ 2\left(\frac{11}{4}\right) + \left(\frac{-3}{2}\right) &= \frac{22}{4} + \frac{-6}{4} = \frac{16}{4} = 4 \end{aligned}$$

2.3. Consistent systems. When a system of equations has at least one solution, it is said to be consistent.

2.4. Unique solutions. When a system of equations has one and only one solution, we say that the solution is unique.

2.5. **Possible sets of solutions for a system of equations.** For a given system of equations.

1. We may have **one and only one** solution. In this case we say we have a **unique** solution.
2. We may have **no** solutions. In this case we say that the system of equations is **inconsistent**.
3. We may have **more than one set of solutions** but the set of solutions is not infinite. We say that the number of solutions is **finite**. This can occur when one of more variables appears in an equation with a power other than one and/or there is a product of variables in one of the equations. These are non-linear equations.
4. We have an **infinite number of solutions** to the system of equations. In this case we say the system is **indeterminate**.

3. METHODS OF FINDING SOLUTIONS FOR EQUATIONS

3.1. Method of substitution.

- a. Pick one of the equations and solve for one of the variables in terms of the remaining variables.
- b. Substitute the result in the remaining equations.
- c. If one equation in one variable results, solve this equation. Otherwise repeat step a until a single equation with a single variable remains.
- d. Find the values of the remaining variables by back-substitution.
- e. Check the solution found.

3.2. Examples of the method of substitution.

3.2.1. *Example 1.* Consider system b in section 1.1 above.

$$2x_1 + 3x_2 = 7 \quad (\text{b.1})$$

$$x_1 - x_2 = -1 \quad (\text{b.2})$$

Rewrite the second equation (b.2) as follows

$$x_1 = x_2 - 1 \quad (\text{b.2a})$$

Now substitute in the first equation (b.1), obtain one equation in one unknown, and solve to obtain

$$\begin{aligned} 2(x_2 - 1) + 3x_2 &= 7 \\ \Rightarrow 2x_2 - 2 + 3x_2 &= 7 \\ \Rightarrow 5x_2 &= 9 \\ \Rightarrow x_2 &= \frac{9}{5} \end{aligned}$$

Now substitute $x_2 = 9/5$ in the second equation to obtain

$$\begin{aligned} x_1 - \frac{9}{5} &= -1 \\ \Rightarrow x_1 &= -1 + \frac{9}{5} = \frac{4}{5} \end{aligned}$$

We can verify the answer as

$$\begin{aligned} 2\left(\frac{4}{5}\right) + 3\left(\frac{9}{5}\right) &= \frac{8}{5} + \frac{27}{5} = \frac{35}{5} = 7 \\ \frac{4}{5} - \frac{9}{5} &= -\frac{5}{5} = -1 \end{aligned}$$

3.2.2. *Example 2.* Consider system d in section 1.1 above.

$$x + y + z = 6 \quad (1d.1)$$

$$3x - 2y + 4z = 9 \quad (1d.2)$$

$$x - y - z = 0 \quad (1d.3)$$

Rewrite the third equation (1d.3) as follows

$$x = y + z \quad (1d.3')$$

Now substitute in the first and second equations to obtain

$$\begin{aligned} (y + z) + y + z &= 6 \\ \Rightarrow 2y + 2z &= 6 \end{aligned} \quad (1d.1.a)$$

$$\begin{aligned} 3(y + z) - 2y + 4z &= 9 \\ \Rightarrow 3y + 3z - 2y + 4z &= 9 \\ \Rightarrow y + 7z &= 9 \end{aligned} \quad (1d.2.a)$$

Now solve the new second equation for y as follows

$$\begin{aligned} y + 7z &= 9 \\ \Rightarrow y &= 9 - 7z \end{aligned} \quad (1d.2.a')$$

Now substitute in the first equation to obtain

$$\begin{aligned} 2y + 2z &= 6 \\ \Rightarrow 2(9 - 7z) + 2z &= 6 \\ \Rightarrow 18 - 14z + 2z &= 6 \\ \Rightarrow 12z &= 12 \\ \Rightarrow z &= 1 \end{aligned} \quad (1d.1b)$$

Now substitute $z = 1$ in the revised second equation to obtain

$$\begin{aligned} y &= 9 - 7z \\ \Rightarrow y &= 9 - 7(1) = 2 \end{aligned} \quad (1d.2b)$$

Then substitute y and z in the revised third equation to obtain

$$\begin{aligned} x &= y + z \\ \Rightarrow x &= 2 + 1 = 3 \end{aligned} \quad (1d.3b)$$

So the answer is $(x, y, z) = (3, 2, 1)$. We can verify the answer as

$$\begin{aligned} 3 + 2 + 1 &= 6 \\ 3(3) - 2(2) + 4(1) &= 9 \\ 3 - 2 - 1 &= 0 \end{aligned}$$

3.2.3. *Example 3.* Consider system c in section 1.1 above.

$$4x + 3y = 24 \quad (1c.1)$$

$$12x + 9y = 36 \quad (1c.2)$$

Rewrite the second equation as follows

$$12x = 36 - 9y$$

$$\Rightarrow x = 3 - \frac{3}{4}y$$

Now substitute in the first equation, obtain one equation in one unknown and solve to obtain

$$4x + 3y = 24$$

$$\Rightarrow 4\left(3 - \frac{3}{4}y\right) + 3y = 24$$

$$\Rightarrow 12 - 3y = 3y = 24$$

$$\Rightarrow 12 = 24$$

Since this leads to a statement which is not true, the system is inconsistent.

3.2.4. *Example 4.* Consider system f in section 1.1 above.

$$x_1 + x_2 = 5 \quad (1f.1)$$

$$2x_1 - 3x_2 = -5 \quad (1f.2)$$

$$4x_1 - x_2 = 5 \quad (1f.3)$$

Solve the first equation for x_1 as a function of x_2 as follows

$$x_1 + x_2 = 5$$

$$\Rightarrow x_1 = 5 - x_2$$

Now substitute in the second and third equations to obtain

$$2x_1 - 3x_2 = -5$$

$$\Rightarrow 2(5 - x_2) - 3x_2 = -5$$

$$\Rightarrow 10 - 2x_2 - 3x_2 = -5$$

$$\Rightarrow 15 - 5x_2 = 0$$

$$4x_1 - x_2 = 5$$

$$\Rightarrow 4(5 - x_2) - x_2 = 5$$

$$\Rightarrow 20 - 4x_2 - x_2 = 5$$

$$\Rightarrow 15 - 5x_2 = 0$$

We can now solve either equation directly as follows.

$$15 - 5x_2 = 0$$

$$\Rightarrow 5x_2 = 15$$

$$\Rightarrow x_2 = 3$$

$$\Rightarrow x_1 = 2$$

We can verify that these values of x_1 and x_2 satisfy all three equations. Any one of the equations in the system is redundant, meaning we can eliminate it and still solve the system.

If we changed the third equation to $4x_1 - x_2 = -5$ we would obtain the following equation after substituting $5 - x_2$ for x_1 .

$$\begin{aligned} 4x_1 - x_2 &= -5 \\ \Rightarrow 4(5 - x_2) - x_2 &= -5 \\ \Rightarrow 20 - 4x_2 - x_2 &= -5 \\ \Rightarrow 25 - 5x_2 &= 0 \end{aligned}$$

Solving the equation we obtain

$$\begin{aligned} 25 - 5x_2 &= 0 \\ \Rightarrow 5x_2 &= 25 \\ \Rightarrow x_2 &= 5 \\ \Rightarrow x_1 &= 0 \end{aligned}$$

We now have inconsistent solutions to the system.

3.2.5. *Example 5.* Consider system a in section 1.2 above.

$$x + y^2 = 5 \tag{2a.1}$$

$$2x + y = 4 \tag{2a.2}$$

Solve the second equation for y as follows

$$\begin{aligned} 2x + y &= 4 \\ \Rightarrow y &= 4 - 2x \end{aligned}$$

Now substitute in the first equation and simplify to obtain

$$\begin{aligned} x + y^2 &= 5 \\ \Rightarrow x + (4 - 2x)^2 &= 5 \\ \Rightarrow x + 16 - 16x + 4x^2 &= 5 \\ \Rightarrow 4x^2 - 15x + 11 &= 0 \\ \Rightarrow ax^2 + bx + c &= 0 \end{aligned}$$

Now solve the equation using the quadratic formula as follows

$$\begin{aligned}
 x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (a \neq 0) \\
 \Rightarrow x &= \frac{-(-15) \pm \sqrt{(-15)^2 - 4(4)(11)}}{2(4)} \\
 &= \frac{15 \pm \sqrt{225 - 176}}{8} \\
 &= \frac{15 \pm \sqrt{49}}{8} \\
 &= \frac{15 \pm 7}{8} \\
 \Rightarrow x &= \frac{22}{8} = \frac{11}{4}, \text{ or } x = 1
 \end{aligned}$$

Substituting $x = 1$ in the modified second equation will give

$$\begin{aligned}
 y &= 4 - 2x \\
 \Rightarrow y &= 4 - 2(1) = 2
 \end{aligned}$$

Substituting $x = 11/4$ in the modified second equation will give

$$\begin{aligned}
 y &= 4 - 2x \\
 \Rightarrow y &= 4 - 2\left(\frac{11}{4}\right) = 4 - \frac{22}{4} \\
 &= \frac{16}{4} - \frac{22}{4} = -\frac{6}{4} = -\frac{3}{2}
 \end{aligned}$$

3.2.6. *Example 6.* Consider system b in section 1.2 above.

$$6x_1^{-1/2}x_2^{1/3} = 4 \quad (1.2b.1)$$

$$4x_1^{1/2}x_2^{-2/3} = 3 \quad (1.2b.2)$$

Rearrange the first equation 1.2b.1 to obtain

$$\begin{aligned}
 x_1^{-1/2}x_2^{1/3} &= \frac{4}{6} = \frac{2}{3} \\
 \Rightarrow x_1^{1/2}x_1^{-1/2}x_2^{1/3} &= \frac{2}{3}x_1^{1/2} \\
 \Rightarrow x_2^{1/3} &= \frac{2}{3}x_1^{1/2} \\
 \Rightarrow x_2 &= \left(\frac{2}{3}\right)^3 (x_1^{1/2})^3 \\
 &= \left(\frac{2}{3}\right)^3 x_1^{3/2}
 \end{aligned} \quad (1.2b.1.a)$$

Rearrange the second equation 1.2b.2 slightly to obtain

$$x_1^{1/2}x_2^{-2/3} = \frac{3}{4} \quad (1.2b.2')$$

Now substitute x_2 from equation 1.2b.1.a into equation 1.2b.2' to obtain

$$\begin{aligned}
 x_1^{1/2} \left(\left(\frac{2}{3} \right)^3 x_1^{3/2} \right)^{-2/3} &= \frac{3}{4} \\
 \Rightarrow x_1^{1/2} \left(\frac{2}{3} \right)^{-2} x_1^{-1} &= \frac{3}{4} \\
 \Rightarrow x_1^{-1/2} &= \frac{3}{4} \left(\frac{2}{3} \right)^2 \\
 \Rightarrow x_1 &= \left(\frac{3}{4} \left(\frac{2}{3} \right)^2 \right)^{-2} && (1.2b.2.a) \\
 &= \left(\frac{3}{2^2} \left(\frac{2}{3} \right)^2 \right)^{-2} \\
 &= 3^{-2} 2^4 2^{-4} 3^4 \\
 \Rightarrow x_1 &= 3^2 = 9
 \end{aligned}$$

Now substitute x_1 from equation 1.2b.2.a into equation 1.2b.1.a to obtain

$$\begin{aligned}
 x_2 &= \left(\frac{2}{3} \right)^3 x_1^{3/2} \\
 &= \left(\frac{2}{3} \right)^3 9^{3/2} \\
 &= \left(\frac{2}{3} \right)^3 (3^2)^{3/2} \\
 &= \left(\frac{2}{3} \right)^3 3^3 \\
 &= 2^3 = 8
 \end{aligned}$$

3.3. Method of elimination.

3.3.1. *Basic idea.* The idea behind the method of elimination is to keep replacing the original equations in the system with equivalent equations until an obvious solution is reached. In effect we keep replacing equations until each equation contains just one variable.

3.3.2. *Principles for obtaining equivalent systems of equations.*

- 1: Changing the order in which the equations are listed produces an equivalent system.
- 2: Suppose that in a system of equations, we multiply both sides of a single equation by a nonzero number (leaving the other equations) unchanged. Then the resulting system of equations is equivalent to the original system.
- 3: Suppose that in a system of equations, we add a multiple of one equation to another equation (leaving the other equations) unchanged. Then the resulting system of equations is equivalent to the original system.

3.3.3. *Proof of principles.*

- 1: This principle is obvious.
- 2: Proof of principle 2

Proof. The key is that multiplying by a nonzero constant k is reversible, because we can also multiply by $1/k$. More specifically, we can argue as follows. If $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, then of course $ka_1x_1 + ka_2x_2 + \cdots + ka_nx_n = kb$. So any solution (x_1, \dots, x_n) to the original system is also a solution to the new system. Moreover, whenever we have a solution to $ka_1x_1 + ka_2x_2 + \cdots + ka_nx_n = kb$, then multiplying the equation by $1/k$ shows that $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$. Thus any solution to the new system is also a solution to the original system. \square

- 3: Proof of principle 3

Proof. Suppose we add k times the equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ to the equation $c_1x_1 + c_2x_2 + \cdots + c_nx_n = d$. If these equations are true, then the sum

$$(ka_1 + c_1)x_1 + (ka_2 + c_2)x_2 + \cdots + (ka_n + c_n)x_n = kb + d$$

is true; that is, any solution to the original system is also a solution to the new system. Could extraneous solutions have entered? Observe that the original system can be recovered from the new system by adding $-k$ times the equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ to

$$(ka_1 + c_1)x_1 + (ka_2 + c_2)x_2 + \cdots + (ka_n + c_n)x_n = kb + d.$$

So the argument of the preceding paragraph can be reapplied (with $-k$ in place of k) to assure us that any solution to the new system was already a solution to the original system. \square

3.3.4. *Steps in the case of a linear system.*

- 1: Take the first variable x_s (that is, the first in the order x_1, x_2, \dots, x_n) having a non-zero coefficient in some equation. Then choose one of the equations in which the coefficient c of x_s is nonzero. We say this equation is the one being used. Place it just above the equations not yet used.
- 2: Multiply the equation being used by $1/c$, so that its new coefficient for x_s is 1.
- 3: To each of the other equations, (both used and unused) in which x_s has a non-zero coefficient h , add $-h$ times the equation being used. This eliminates x_s from every equation except the one being used. We now say that the variable x_s has been used.
- 4: Now take the next variable, x_t , having a nonzero coefficient in some unused equation. Following steps 1, 2, and 3, eliminate it from all other equations, including those already used. Again the variable x_t and the equation we have worked with in this step is called used. Step 4 is then repeated until there is no variable with a nonzero coefficient in an unused equation.

If there are solutions to a system of linear equations, they will be obtained from this procedure. If no solutions exist, a contradiction will be obtained.

3.3.5. Examples.

Example 1: Consider system a in section 1.1 above.

$$2x + y = 5 \quad (\text{a.1})$$

$$-4x + 6y = -2 \quad (\text{a.2})$$

Let x be the variable chosen and let the first equation be the equation chosen. It is already at the top of the system. Multiply the first equation by $\frac{1}{2}$ and then rewrite the entire system as follows

$$x + \frac{y}{2} = \frac{5}{2}$$

$$-4x + 6y = -2$$

Multiply the first equation 4 and add it to the second equation. This gives

$$\begin{array}{r} 4x + \frac{4y}{2} = \frac{20}{2} \\ \Rightarrow 4x + 2y = 10 \\ -4x + 6y = -2 \\ \hline 8y = 8 \end{array}$$

Now rewrite the system as follows

$$\begin{array}{r} x + \frac{y}{2} = \frac{5}{2} \\ 8y = 8 \end{array}$$

Now choose the second equation and the second variable. Multiply it by $1/8$ and then rewrite the system as

$$\begin{array}{r} x + \frac{y}{2} = \frac{5}{2} \\ y = 1 \end{array}$$

Multiply the second equation by $-1/2$ and add it to the first equation

$$\begin{array}{r} x + \frac{y}{2} = \frac{5}{2} \\ -\frac{1}{2}y = -\frac{1}{2} \\ \hline x = \frac{4}{2} = 2 \end{array}$$

The reduced system is now given by

$$x = 2$$

$$y = 1$$

Example 2: Consider system d in section 1.1 above.

$$x + y + z = 6 \quad (\text{d.1})$$

$$3x - 2y + 4z = 9 \quad (\text{d.2})$$

$$x - y - z = 0 \quad (\text{d.3})$$

Choose the first variable and the first equation. Since the coefficient of x is 1, we can proceed directly. We will add (-3) times it to the second equation and (-1) times it to the third equation. We will first use (-3) as the multiplier and add the first equation to the second.

$$\begin{array}{rcccc} -3x & -3y & -3z & = & -18 \\ 3x & -2y & +4z & = & 9 \\ \hline & -5y & +z & = & -9 \end{array}$$

Now rewrite the system as follows.

$$\begin{array}{rcccc} x & +y & +z & = & 6 \\ & -5y & +z & = & -9 \\ x & -y & -z & = & 0 \end{array}$$

Now write the system using (-1) as the multiplier for the first equation and add to the third equation as follows

$$\begin{array}{rcccc} -x & -y & -z & = & -6 \\ x & -y & -z & = & 0 \\ \hline & -2y & -2z & = & -6 \end{array}$$

Now rewrite the system as follows.

$$\begin{array}{rcccc} x + & y & +z & = & 6 \\ & -5y & +z & = & -9 \\ & -2y & -2z & = & -6 \end{array}$$

The variable x is now eliminated from all but the first equation. Now choose the second equation and y as the variable. Since its coefficient is -5 , we multiply by $1/5$ and add it to the first equation as follows.

$$\begin{array}{rcccc} x + & y & +z & = & 6 \\ & - & y & + \frac{1}{5}z & = -\frac{9}{5} \\ \hline x & & + \frac{6}{5}z & = & \frac{21}{5} \end{array}$$

The system is now given by

$$\begin{array}{rcccc} x & & + \frac{6}{5}z & = & +\frac{21}{5} \\ & - & y & + \frac{1}{5}z & = -\frac{9}{5} \\ & - & 2y & - 2z & = -6 \end{array}$$

Now multiply the second equation by -2 and add to the third equation to eliminate y from it. This will give

$$\begin{array}{rcl}
 2y & -\frac{2}{5}z & = \frac{18}{5} \\
 -2y & -2z & = -6 \\
 \hline
 & -\frac{12}{5}z & = -\frac{12}{5}
 \end{array}$$

The system is now given by

$$\begin{array}{rcl}
 x & +\frac{6}{5}z & = \frac{21}{5} \\
 -y & +\frac{1}{5}z & = -\frac{9}{5} \\
 & -\frac{12}{5}z & = -\frac{12}{5}
 \end{array}$$

The variable y now only appears in the second equation. Now use the third equation and z. Since the coefficient of z is -12/5 we multiply by -5/12. This gives for the entire system

$$\begin{array}{rcl}
 x & +\frac{6}{5}z & = +\frac{21}{5} \\
 -y & +\frac{1}{5}z & = -\frac{9}{5} \\
 & z & = 1
 \end{array}$$

First multiply the last equation by -6/5 and add to the first one. This will give

$$\begin{array}{rcl}
 x & +\frac{6}{5}z & = \frac{21}{5} \\
 & -\frac{6}{5}z & = -\frac{6}{5} \\
 \hline
 x & & = \frac{15}{5} \\
 \Rightarrow x & & = 3
 \end{array}$$

Now multiply the third equation by -1/5 and add to the second equation.

$$\begin{array}{rcl}
 -y & +\frac{1}{5}z & = -\frac{9}{5} \\
 & -\frac{1}{5}z & = -\frac{1}{5} \\
 \hline
 -y & & = -\frac{10}{5} \\
 \Rightarrow y & & = 2
 \end{array}$$

Putting it all together we obtain

$$\begin{array}{l}
 x = 3 \\
 y = 2 \\
 z = 1
 \end{array}$$

Example 3: Consider system c in section 1.1 above.

$$4x + 3y = 24 \quad (\text{c.1})$$

$$12x + 9y = 36 \quad (\text{c.2})$$

Choose the first variable and the first equation. Since the coefficient of x is 4, divide the first equation by 4 and rewrite the system as

$$x + \frac{3}{4}y = 6$$

$$12x + 9y = 36$$

Now multiply the first equation by -12 and add to the second equation.

$$\begin{array}{r} -12x - 9y = -72 \\ 12x + 9y = 36 \\ \hline 0x + 0y = -36 \end{array}$$

It is clear that we have an inconsistent system.

Example 4: Consider the following system

$$4x + 2y = 12 \quad (\text{q.1})$$

$$6x + 3y = 18 \quad (\text{q.2})$$

Choose the first variable and the first equation. Since the coefficient of x is 4, divide the first equation by 4 and rewrite the system as

$$x + \frac{1}{2}y = 3$$

$$6x + 3y = 18$$

Now multiply the first equation by -6 and add to the second equation.

$$\begin{array}{r} -6x - 3y = -18 \\ 6x + 3y = 18 \\ \hline 0x + 0y = 0 \end{array}$$

We now have an equation stating that $0x + 0y = 0$. This is a true statement, but provides no information about the values of x and y that solve the system. Solve the second equation for x to obtain

$$\begin{aligned} 6x + 3y &= 18 \\ \Rightarrow 6x &= 18 - 3y \\ \Rightarrow x &= 3 - \frac{1}{2}y \end{aligned} \quad (\text{q}')$$

Any x satisfying equation (q') will satisfy both equations in q . There are infinite solutions to system q , one for every possible value of y among the real numbers.

Example 5: Consider the following system of equations.

$$x_1 \qquad +2x_2 \qquad -2x_3 = \qquad -2 \qquad (r.1)$$

$$2x_1 \qquad +5x_2 \qquad -3x_3 = \qquad -4 \qquad (r.2)$$

$$-x_1 \qquad +3x_2 \qquad +8x_3 = \qquad 3 \qquad (r.3)$$

Choose the first variable and the first equation. Since the coefficient of x_1 is 1, we can proceed directly. We will add (-2) times it to the second equation and (1) times it to the third equation. We will first use (-2) as the multiplier and add the first equation to the second.

$$\begin{array}{r} -2x_1 \quad -4x_2 \quad +4x_3 = \quad 4 \\ 2x_1 \quad +5x_2 \quad -3x_3 = \quad -4 \\ \hline \qquad \qquad +x_2 \quad +x_3 = \quad 0 \end{array}$$

Now rewrite the system as follows.

$$\begin{array}{r} x_1 \quad +2x_2 \quad -2x_3 = \quad -2 \\ \qquad +x_2 \quad +x_3 = \quad 0 \\ -x_1 \quad +3x_2 \quad +8x_3 = \quad 3 \end{array}$$

Now add the first equation to the third equation as follows

$$\begin{array}{r} x_1 \quad +2x_2 \quad -2x_3 = \quad -2 \\ -x_1 \quad +3x_2 \quad +8x_3 = \quad 3 \\ \hline \qquad \qquad +5x_2 \quad +6x_3 = \quad 1 \end{array}$$

Now rewrite the system as follows.

$$\begin{array}{r} x_1 \quad +2x_2 \quad -2x_3 = \quad -2 \\ \qquad +x_2 \quad +x_3 = \quad 0 \\ \qquad +5x_2 \quad +6x_3 = \quad 1 \end{array}$$

The variable x_1 is now eliminated from all but the first equation. Now choose the second equation and x_2 as the variable. Since its coefficient is 1 we multiply by -5 and add it to the third equation as follows.

$$\begin{array}{r} -5x_2 \quad -5x_3 = \quad 0 \\ +5x_2 \quad +6x_3 = \quad 1 \\ \hline \qquad \qquad +x_3 = \quad 1 \end{array}$$

The system is now given by

$$\begin{array}{rclcl} x_1 & +2x_2 & -2x_3 & = & -2 \\ & +x_2 & +x_3 & = & 0 \\ & & +x_3 & = & 1 \end{array}$$

Now multiply the third equation by -1 and add to the second equation. This will give

$$\begin{array}{rclcl} & +x_2 & +x_3 & = & 0 \\ & & -x_3 & = & -1 \\ - - - & - - - & - - - & = & - - \\ & +x_2 & & = & -1 \end{array}$$

The system is now given by

$$\begin{array}{rclcl} x_1 & +2x_2 & -2x_3 & = & -2 \\ & +x_2 & & = & -1 \\ & & +x_3 & = & 1 \end{array}$$

Now multiply the third equation by 2 and add to the first equation.

$$\begin{array}{rclcl} x_1 & +2x_2 & -2x_3 & = & -2 \\ & & +2x_3 & = & 2 \\ - - - & - - - & - - - & = & - - \\ x_1 & +2x_2 & & = & 0 \end{array}$$

The system is now given by

$$\begin{array}{rclcl} x_1 & +2x_2 & & = & 0 \\ & +x_2 & & = & -1 \\ & & +x_3 & = & 1 \end{array}$$

Now multiply the second equation by -2 and add to the first equation. This will give

$$\begin{array}{rclcl} x_1 & +2x_2 & & = & 0 \\ & -2x_2 & & = & 2 \\ - - - & - - - & - - - & = & - - \\ x_1 & & & = & 2 \end{array}$$

The system is now given by

$$\begin{array}{rclcl} x_1 & & & = & 2 \\ & +x_2 & & = & -1 \\ & & +x_3 & = & 1 \end{array}$$

and we are done.

