PRODUCTION FUNCTIONS

1. ALTERNATIVE REPRESENTATIONS OF TECHNOLOGY

The technology that is available to a firm can be represented in a variety of ways. The most general are those based on correspondences and sets.

1.1. **Technology Sets.** The technology set for a given production process is defined as

$$T = \{(x, y): \ x \in R_+^n, \ y \in R_+^m : x \text{ can produce } y\}$$

where x is a vector of inputs and y is a vector of outputs. The set consists of those combinations of x and y such that y can be produced from the given x.

1.2. **The Output Correspondence and the Output Set.**

1.2.1. *Definitions.* It is often convenient to define a production correspondence and the associated output set.

1: The output correspondence P, maps inputs $x \in R_+^n$ into subsets of outputs, i.e., P: $R_+^n \to 2^{R_+^m}$. A correspondence is different from a function in that a given domain is mapped into a set as compared to a single real variable (or number) as in a function.

2: The output set for a given technology, P(x), is the set of all output vectors $y \in R_+^m$ that are obtainable from the input vector $x \in R_+^n$. P(x) is then the set of all output vectors $y \in R_+^m$ that are obtainable from the input vector $x \in R_+^n$. We often write P(x) for both the set based on a particular value of x, and the rule (correspondence) that assigns a set to each vector x.

1.2.2. *Relationship between P(x) and T(x,y).*

$$P(x) = (y : (x, y) \in T)$$

1.2.3. *Properties of P(x).*

P.1a: Inaction and No Free Lunch. $0 \in P(x) \ \forall \ x \in R_+^n$.
P.1b: $y \notin P(0)$, $y > 0$.
P.2: Input Disposability. $\forall \ x \in R_+^n$, $P(x) \subseteq P(\theta x)$, $\theta \geq 1$.
P.2.S: Strong Input Disposability. $\forall \ x, x' \in R_+^n$, $x' \geq x \Rightarrow P(x) \subseteq P(x')$.
P.3: Output Disposability. $\forall \ x \in R_+^n$, $y \in P(x)$ and $0 \leq \lambda \leq 1 \Rightarrow \lambda y \in P(x)$.
P.3.S: Strong Output Disposability. $\forall \ x \in R_+^n$, $y \in P(x) \Rightarrow y' \in P(x)$, $0 \leq y' \leq y$.
P.4: Boundedness. P(x) is bounded for all $x \in R_+^n$.
P.5: T is a closed set P: $R_+^n \to 2^{R_+^m}$ is a closed correspondence, i.e., if $[x^\ell \to x^0, y^\ell \to y^0$ and $y^\ell \in P(x^\ell), \ \forall \ \ell]$ then $y^0 \in P(x^0)$.
P.6: Attainability. If $y \in P(x)$, $y \geq 0$ and $x \geq 0$, then $\forall \ \theta \geq 0, \exists \ \lambda_\theta \geq 0$ such that $\theta y \in P(\lambda_\theta x)$.

P.7: P(x) is convex

P(x) is a convex set for all x $\in R_+^n$. This is equivalent to the correspondence V:$\Re_+^n \rightarrow 2^{\Re_+^m}$ being quasiconcave.

P.8: P is quasi-concave.

The correspondence P is quasi-concave on R_+^n which means $\forall$ x, x′ $\in R_+^n$, $0 \le \theta \le 1$, P(x) $\cap$ P(x′) $\subseteq$ P(θx + (1-θ)x′). This is equivalent to V(y) being a convex set.

P.9: Convexity of T. P is concave on R_+^n which means $\forall$ x, x′ $\in R_+^n$, $0 \le \theta \le 1$, θP(x)+(1-θ)P(x′) $\subseteq$ P(θx + (1-θ)x′)

1.3. **The Input Correspondence and Input (Requirement) Set.**

1.3.1. *Definitions.* Rather than representing a firm's technology with the technology set T or the production set P(x), it is often convenient to define an input correspondence and the associated input requirement set.

1: The input correspondence maps outputs y $\in R_+^m$ into subsets of inputs, V: $R_+^m \rightarrow 2^{R_+^n}$. A correspondence is different from a function in that a given domain is mapped into a set as compared to a single real variable (or number) as in a function.

2: The input requirement set V(y) of a given technology is the set of all combinations of the various inputs x $\in R_+^n$ that will produce at least the level of output y $\in R_+^m$. V(y) is then the set of all input vectors x $\in R_+^n$ that will produce the output vector y $\in R_+^m$. We often write V(y) for both the set based on a particular value of y, and the rule (correspondence) that assigns a set to each vector y.

1.3.2. *Relationship between V(y) and T(x,y).*

$$V(y) = (x: (x, y) \in T)$$

1.4. **Relationships between Representations: V(y), P(x) and T(x,y).** The technology set can be written in terms of either the input or output correspondence.

$$T = \{(x, y) : x \in R_+^n, y \in R_+^m, \text{such that x will produce y}\} \qquad (1a)$$

$$T = \{(x, y) \in R_+^{n+m} : y \in P(x), x \in R_+^n\} \qquad (1b)$$

$$T = \{(x, y) \in R_+^{n+m} : x \in V(y), y \in R_+^m\} \qquad (1c)$$

We can summarize the relationships between the input correspondence, the output correspondence, and the production possibilities set in the following proposition.

Proposition 1. $y \in P(x) \Leftrightarrow x \in V(y) \Leftrightarrow (x,y) \in T$

2. PRODUCTION FUNCTIONS

2.1. **Definition of a Production Function.** To this point we have described the firm's technology in terms of a technology set T(x,y), the input requirement set V(y) or the output set P(x). For many purposes it is useful to represent the relationship between inputs and outputs using a mathematical function that maps vectors of inputs into a single measure of output. In the case where there is a single

output it is sometimes useful to represent the technology of the firm with a mathematical function that gives the maximum output attainable from a given vector of inputs. This function is called a production function and is defined as
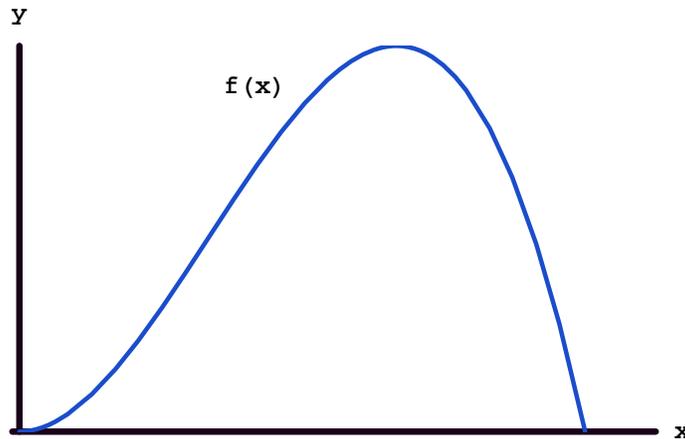
$$
\begin{aligned}
f(x) &= \max_{y} [y : (x, y) \in T] \\
&= \max_{y} [y : x \in V(y)] \\
&= \max_{y \in P(x)} [y]
\end{aligned}
\tag{2}
$$

Once the optimization is carried out we have a numerically valued function of the form

$$
y = f(x_1, x_2, \ldots, x_n)
\tag{3}
$$

Graphically we can represent the production function in two dimensions as in figure 1.

FIGURE 1. Production Function

In the case where there is one output, one can also think of the production function as the boundary of P(x), i.e., f(x) = Eff P(x).

2.2. **Existence and the Induced Production Correspondence.** Does the production function exist. If it exists, is the output correspondence induced by it the same as the original output correspondence from which f was derived? What properties does f(x) inherit from P(x)?

- **a:** To show that production function exists and is well defined, let $x \in R_+^n$. By axiom P.1a, $P(x) \neq \emptyset$. By axioms P.4 and P.5, P(x) is compact. Thus P(x) contains a maximal element and f(x) is well defined. NOte that only these three of the axioms on P are needed to define the production function.
- **b:** The output correspondence induced by f(x) is defined as follows

$$
P_f(x) = [y \in R_+ : f(x) \geq y], \ x \in R_+^n
\tag{4}
$$

This gives all output levels y that can be produced by the input vector x. We can show that this induced correspondence is equivalent to the output correspondence that produced f(x). We state this in a proposition.

Proposition 2. $P_f(x) = P(x), \forall x \in R_+^n$.

Proof. Let $y \in P_f(x), x \in R_+^n$. By definition, $y \leq f(x)$. This means that $y \leq \max\{z: z \in P(x)\}$. Then by P.3.S, $y \in P(x)$. Now show the other way. Let $y \in P(x)$. By the definition of f, $y \leq \max\{z: z \in P(x)\} = f(x)$. Thus $y \in P_f(x)$. □

Properties P.1a, P.3, P.4 and P.5 are sufficient to yield the induced production correspondence.

2.2.1. *Relationship between P(x) and f(x).* We can summarize the relationship between P and f with the following proposition:

Proposition 3. $y \in P(x) \Leftrightarrow f(x) \geq y, \forall x \in R_+^n$

2.3. **Examples of Production Functions.**

2.3.1. *Production function for corn.* Consider the production technology for corn on a per acre basis. The inputs might include one acre of land and various amounts of other inputs such as tillage operations made up of tractor and implement use, labor, seed, herbicides, pesticides, fertilizer, harvesting operations made up of different combinations of equipment use, etc. If all but the fertilizer are held fixed, we can consider a graph of the production relationship between fertilizer and corn yield. In this case the production function might be written as

$$y = f(\text{land, tillage, labor, seed, fertilizer, } \dots) \tag{5}$$

2.3.2. *Cobb-Douglas production function.* Consider a production function with two inputs given by $y = f(x_1, x_2)$. A Cobb-Douglas [4] [5] represention of technology has the following form.

$$y = A x_1^{\alpha_1} x_2^{\alpha_2}$$
$$= 5 \, x_1^{\frac{1}{3}} x_2^{\frac{1}{4}} \tag{6}$$

Figure 2 is a graph of this production function.

Figure 3 shows the contours of this function.

With a single output and input, a Cobb-Douglas production function has the shape shown in figure 4.

2.3.3. *Polynomial production function.* We often approximate a production function using polynomials. For the case of a single input, a cubic production function would take the following form.

$$y = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$$
$$= 10 \, x + 20 \, x^2 - 0.60 \, x^3 \tag{7}$$

The cubic production function in equation 7 is shown in figure 5.
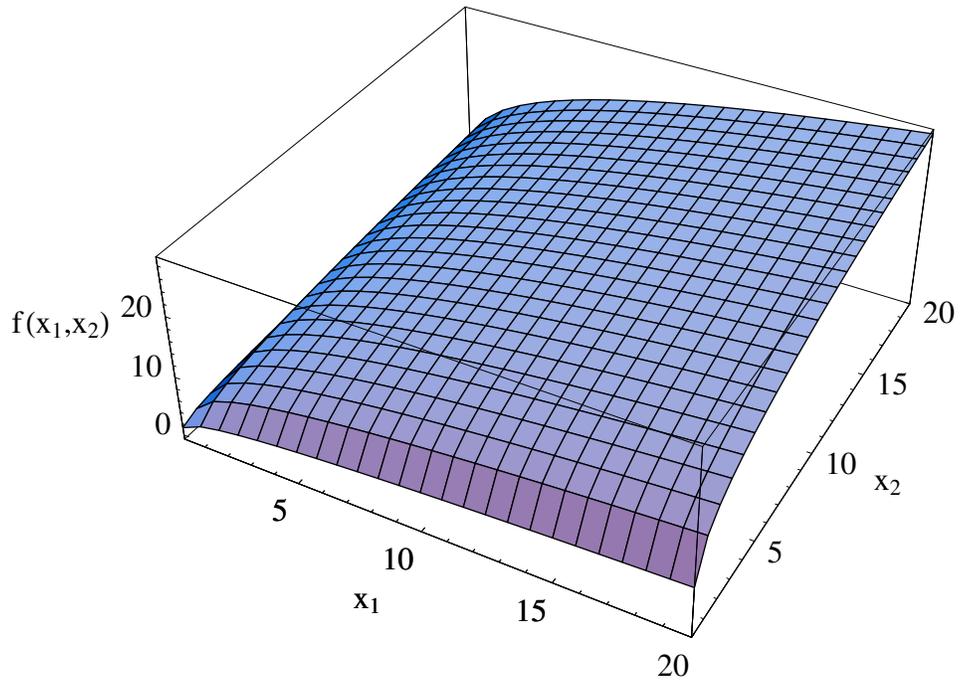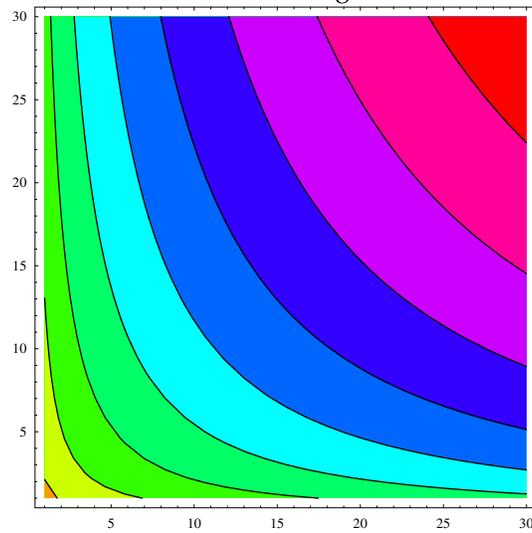
FIGURE 2. Cobb-Douglas Production Function

FIGURE 3. Contours of a Cobb-Douglas Production Function

Notice that the function first rises at an increasing rate, then increases at a decreasing rate and then begins to fall until it reaches zero.

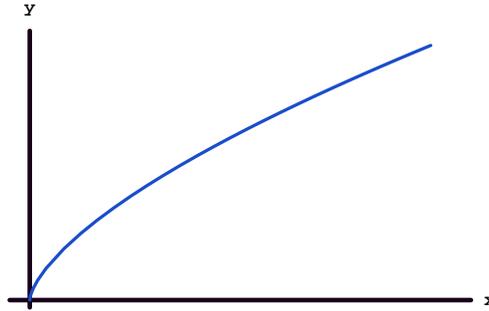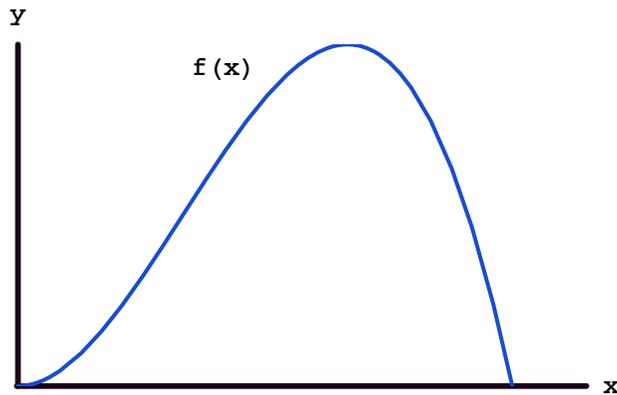FIGURE 4. Cobb-Douglas Production Function with One Input

FIGURE 5. Cubic Production Function

2.3.4. *Constant elasticity of substitution (CES) production function.* An early alternative to the Cobb-Douglas production function is the constant elasticity of substitution (CES) production function [1]. While still being quite tractable, with a minimum of parameters, it is more flexible than the Cobb-Douglas production function. For the case of two inputs, the CES production function takes the following form.

$$
\begin{aligned}
y &= A \left[\delta_1 \, x_1^{-\rho} \, + \, \delta_2 \, x_2^{-\rho} \right]^{\frac{-1}{\rho}} \\
&= 5 \left[0.6 \, x_1^{-2} \, + \, 0.2 \, x_2^{-2} \right]^{\frac{-1}{2}}
\end{aligned}
\tag{8}
$$

The CES production function in equation 8 is shown in figure 6. The production contours of the production function in equation 8 are shown in figure 7. If we change ρ to 0.2, the CES contours are as in figure 8.

2.3.5. *Translog production function.* An alternative to the Cobb-Douglas production function is the translog production function.
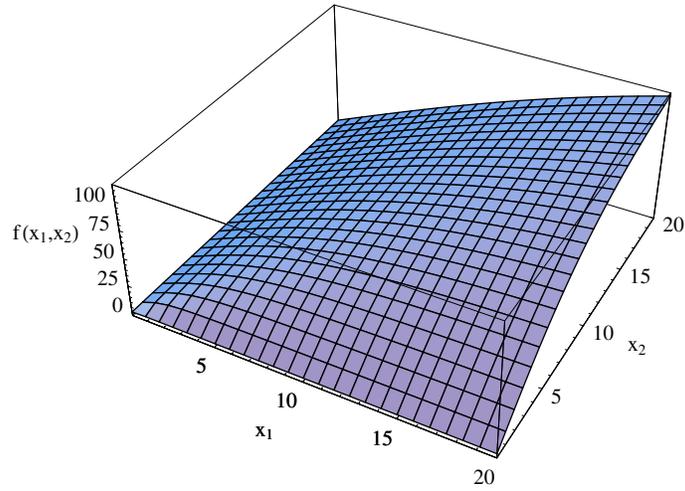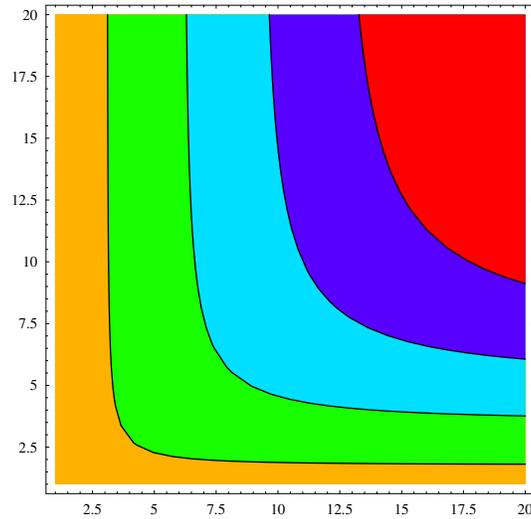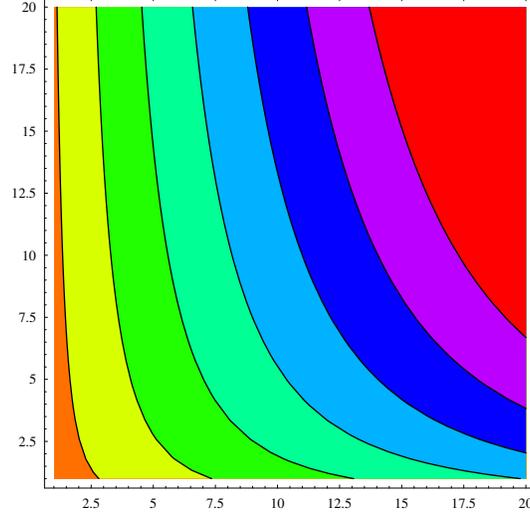
FIGURE 6. CES Production Function

FIGURE 7. CES Production Function Contours - ρ=2

$$ln\, y = \alpha_1 \, ln\, x_1 \, + \, \alpha_2 \, ln\, x_2 \, + \, \beta_{11} \, ln\, x_1^2 \, + \, \beta_{12} \, ln\, x_1 \, ln\, x_2 \, + \, \beta_{22} \, ln\, x_2^2$$
$$= \frac{1}{3} \, ln\, x_1 \, + \, \frac{1}{10} \, ln\, x_2 \, - \, \frac{2}{100} \, ln\, x_1^2 \, + \, \frac{1}{10} \, ln\, x_1 \, ln\, x_2 \, - \, \frac{2}{10} \, ln\, x_2^2 \tag{9}$$

The translog can also be written with y as compared to $ln\, y$ on the left hand side.

FIGURE 8. CES Production Function Contours - ρ=0.05

$$
\begin{aligned}
y \; &= A \, , x_1^{\alpha_1} \, x_2^{\alpha_2} \, e^{\beta_{11} \, ln, \, x_1^2 \, + \, \beta_{12} \, ln \, x_1 \, ln \, x_2 \, + \, \beta_{22} \, ln \, x_2^2} \\
&= x_1^{1/3} \, x_2^{1/10} \, e^{-.02 \, ln \, x_1^2 \, + \, 0.1 \, ln \, x_1 \, ln \, x_2 \, - \, 0.2 \, ln \, x_2^2}
\end{aligned}
\tag{10}
$$

Figure 9 shows the translog function from equation 9 while figure 10 shows the contours of the translog function.

2.4. **Properties of the Production Function.** We can deduce a set of properties on f that are equivalent to the properties on P in the sense that if a particular set holds for P, it implies a particular set on f and vice versa.

2.4.1. *f.1 Essentiality.* f(0) = 0.

2.4.2. *f.1.S Strict essentiality.* $f(x_1, x_2, \ldots, 0, \ldots, x_n) = 0$ for all x_i.

2.4.3. *f.2 Monotonicity.* $\forall \, x \in R_+^n, f(\theta x) \geq f(x), \theta \geq 1$.

2.4.4. *f.2.S Strict monotonicity.* $\forall \, x, x' \in R_+^n$, if $x \geq x'$ then $f(x) \geq f(x')$.

2.4.5. *f.3 Upper semi-continuity.* f is upper semi-continuous on R_+^n.

2.4.6. *f.3.S Continuity.* f is continuous on R_+^n.

2.4.7. *f.4 Attainability.* If $f(x) > 0, f(\lambda x) \to +\infty$ as $\lambda \to +\infty$.

2.4.8. *f.5 Quasi-concavity.* f is quasi-concave on R_+^n.

2.4.9. *f.6 Concavity.* f is concave on R_+^n.

2.5. **Discussion of the Properties of the Production Function.**
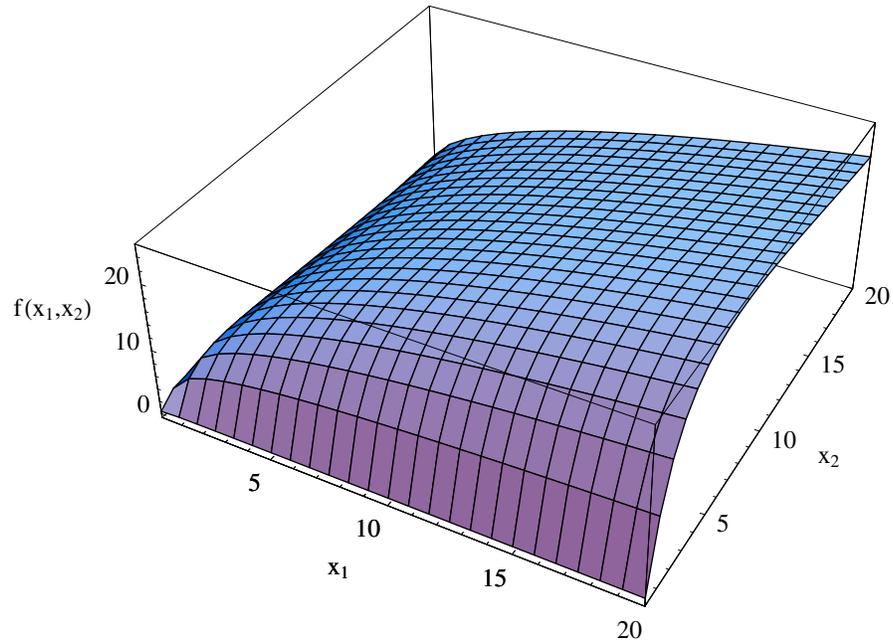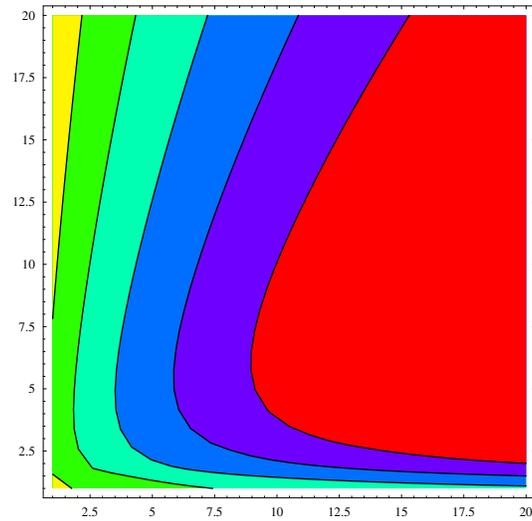
FIGURE 9. Translog Production Function

FIGURE 10. Contours of Translog Production Function

2.5.1. *f.1 Essentiality.* f(0) = 0.

This assumption is sometimes called essentiality. It says that with no inputs, there is no output.

2.5.2. *f.1.S Strict essentiality.* $f(x_1, x_2, \ldots, 0, \ldots, x_n) = 0$ for all x_i.

This is called strict essentiality and says that some of each input is needed for a positive output. In this case the input requirement set doesn't touch any axis. Consider as an example of strict essentiality the Cobb-Douglas function.

$$y \;=\; A\, x_1^{\alpha_1}\, x_2^{\alpha_2} \tag{11}$$

Another example is the Generalized Leontief Function with no linear terms

$$y = \beta_{11}x_1 \;+\; 2\beta_{12}\, x_1\, x_2 \;+\; \beta_{22}\, x_2 \tag{12}$$

2.5.3. *f.2 Monotonicity.* $\forall\, x \in R_+^n$, $f(\theta x) \geq f(x)$, $\theta \geq 1$.

This is a monotonicity assumption that says with a scalar expansion of x, output cannot fall. There is also a strong version.

2.5.4. *f.2 Strict monotonicity.* $\forall\, x, x'$ *in* R_+^n , if $x \geq x'$ then $f(x) \geq f(x')$.

Increasing one input cannot lead to a decrease in output.

2.5.5. *f.3 Upper semi-continuity.* f is upper semi-continuous on R_+^n.

The graph of the production function may have discontinuities, but at each point of discontinuity the function will be continuous from the right. The property of upper semi-continuity is a direct result of the fact that the output and input correspondences are closed. In fact, it follows directly from the input sets being closed.

2.5.6. *f.3.S Continuity.* f is continuous on R_+^n.

We often make the assumption that f is continuous so that we can use calculus for analysis. We sometimes additionally assume the f is continuously differentiable.

2.5.7. *f.4 Attainability.* If $f(x) > 0$, $f(\lambda x) \to +\infty$ as $\lambda \to +\infty$.

This axiom states that there is always a way to exceed any specified output rate by increasing inputs enough in a proportional fashion.
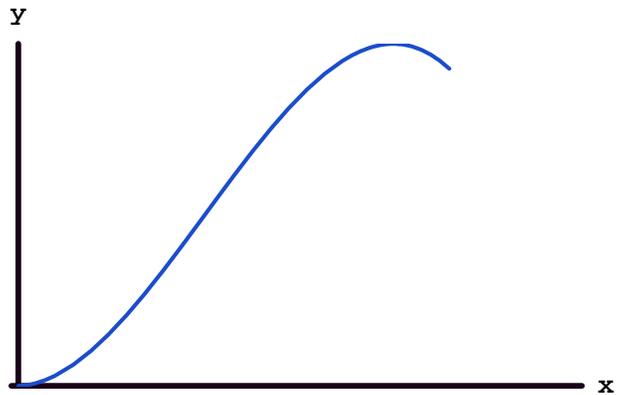
2.5.8. *f.5 Quasi-concavity.* f is quasi-concave on R_+^n.

If a function is quasi-concave then

$$f(x) \;\geq\; f(x^0) \;\Rightarrow\; f(\lambda\, x + (1 - \lambda)\, x^0) \;\geq\; f(x^0) \tag{13}$$

If V(y) is convex then f(x) is quasi-concave because V(y) is an upper contour set of f. This also follows from quasiconcavity of P(x). Consider for example the traditional three stage production function in figure 11. It is not concave, but it is quasi-concave.

FIGURE 11. Quasi-concave Production Function

If the function f is quasi-concave the upper contour or isoquants are convex. This is useful in problems of cost minimization as can be seen in figure 12.

FIGURE 12. Convex Lower Boundary of Input Requirement Set
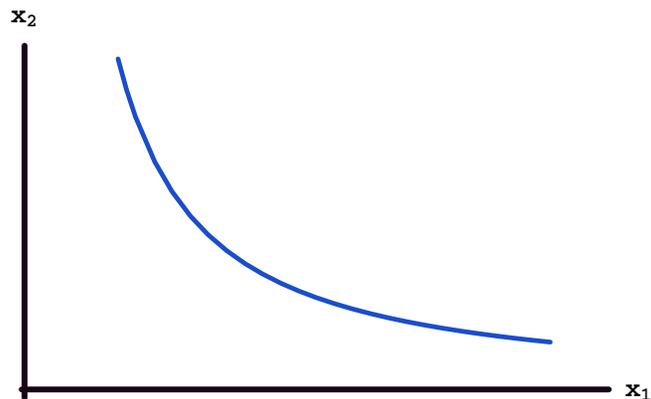
2.5.9. *f.6 Concavity.* f is concave on R_+^n.
    If a function is concave then

$$f(\lambda x \ + \ (1 - \lambda) x') \ \geq \ \lambda f(x) \ + \ (1 - \lambda) f(x') \tag{14}$$

Concavity of f follows from P.9 (V.9) or the overall convexity of the output and input correspondences. This means the level sets are not only convex for a given level of output or input but that the overall correspondence is convex. Contrast the traditional three stage production function with a Cobb-Douglas one. Concavity is implied by the function lying above the chord as can be seen in figure 13 or below the tangent line as in figure 14.

2.6. **Equivalence of Properties of P(x) and f(x).** The properties (f.1 - f.6) on f(x) can be related to specific properties on P(x) and vice versa. Specifically the following proposition holds.

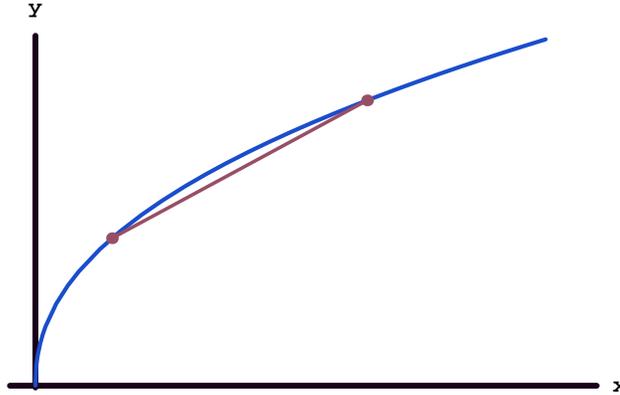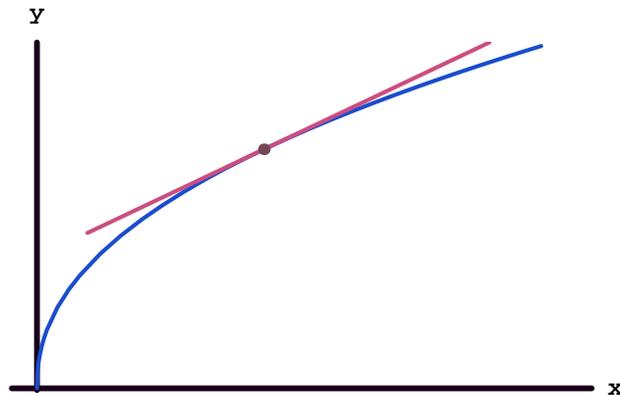FIGURE 13. Concavity Implies that a Chord Lies below the Function

FIGURE 14. A Concave Function Lies Below the Tangent Line

Proposition 4. *The output correspondence P: $R_+^n \to 2^{R_+}$ satisfies P.1 - P.6 iff the production function f : $R_+^n \to R_+$ satisfies f.1 - f.4. Furthermore P.8 $\Leftrightarrow$ f.5 and P.9 $\Leftrightarrow$ f.6.*

Proofs of some of the equivalencies between properties of P(x) and f(x).

2.6.1. *P.1 $\Leftrightarrow$ f.1.* P.1a states that $0 \in P(x) \ \forall \ x \in R_+^n$; $y \notin P(0)$, $y > 0$. Let $x \in R_+^n$ so that $0 \in P(x)$. Then by Proposition 3 $f(x) \geq 0$. Now if $y > 0$ then $y \notin P(0)$ by P.1b. So $y > f(0)$. But $f(x) \geq 0$ by Proposition 3. Thus let $y \to 0$ to obtain $f(0) = 0$.

Now assume that f.1 holds. By Proposition 3 it is obvious that $0 \in P(x)$, $\forall \ x \in R_+^n$. Now compute $P(0) = [y \in R_+: f(0) \geq y$]. This is the empty set unless $y = 0$. So if $y > 0$, then $y \notin P(0)$.

2.6.2. *P.2 $\Leftrightarrow$ f.2.* P.2 states that $\forall \ x \in R_+^n$, $P(x) \subseteq P(\theta x), \theta \geq 1$. Consider the definition of f(x) and f(θx) given by

$$f(x) \ = \ \max \left[y \ \in \ R_+ \ : \ y \ \in \ P(x) \right] x \ \in \ R_+^n$$
$$f(\theta x) \ = \ \max \left[y \ \in \ R_+ \ : \ y \ \in \ P(\theta x) \right] x \ \in \ R_+^n$$

Now because $P(x) \subseteq P(\theta x)$ for $\theta \geq 1$ it is clear that the maximum over the second set must be larger than the maximum over the first set.

To show the other way remember that if $y \in P(x)$ then $f(x) \geq y$. Now assume that $f(\theta x) \geq f(x) \geq y$. This implies that $y \in P(\theta x)$ which implies that $P(x) \subseteq P(\theta x)$.

2.6.3. *Definition of $f \to P.3$.* Remember that P.3 states $\forall\, x \in \Re_+^n$, $y \in P(x)$ and $0 \leq \lambda \leq 1 \Rightarrow \lambda y \in P(x)$. So consider an input vector x and an output level y such that $f(x) \geq y$. Then consider a value of λ such that $0 \leq \lambda \leq 1$. Given the restrction on λ, $y \geq \lambda y$. But by Proposition 3 which follows from the definition of the producion function in equation 2 and Proposition 2, $\lambda y \in P(x)$.

2.6.4. *F.3 $\to$ P.4.* Recall that P.4 is that P(x) is bounded for all x $\in$ R$^n{}_+$. Let $x \in \Re_+^n$. The set

$$M(x) \;=\; \{u \in \Re_+^n : u \leq x\} \tag{15}$$

is compact as it is closed and bounded. F.3 says that f(x) is upper-semicontinuous, thus the maximum

$$f(u^*) \;=\; max\{f(u) : \; u \in M(x)\} \;\geq\; f(x) \tag{16}$$

$u^* \in M(x)$ exists. The closed interval [0,f(x)] is a subset of the closed interval [0,f(u^*)], i.e, $[0, f(x)] \subseteq [0, f(u^*)]$. Therefore P(x) = [0,f(x)] is bounded.

2.6.5. *P.4 $\to$ F.3.* Recall that f(x) is upper semi-continuous at x^0 iff $\lim \sup_{n \to \infty} f(x^n) \leq f(x^0)$ for all sequences $x^n \to x^0$.

Consider a sequence $\{x_n\} \to x^0$. Let $y^n \equiv f(x^n)$. Now suppose that $\lim \sup_{n \to \infty} y^n = \bar{y} \geq f(x^0)$. Then $\{y^n\} \to y^0 \geq f(x^0)$ because the maximum value of the sequence $\{y^n\}$ is greater than $f(x^0)$. Because $P : \Re_+^n \to 2^{\Re_+}$ is a closed correspondence, $y^0 \in [0, f(x^0)]$ and $y^0 \leq f(x^0)$, a contradiction. Thus $\lim \sup_{n \to \infty} f(x^n) \leq f(x^0)$.

2.6.6. *Other equivalencies.* One can show that the following equivalences also hold.
   a: P.6 $\Rightarrow$ f.4
   b: P.7 follows from the definition of P(x) in terms of f in equation 4.

2.7. **Marginal and Average Measures of Production.**

2.7.1. *Marginal product (MP).* The firm is often interested in the effect of additional inputs on the level of output. For example, the field supervisor of an irrigated crop may want to know how much crop yield will rise with an additional application of water during a particular period or a district manager may want to know what will happen to total sales if she adds another salesperson and rearranges the assigned areas. For small changes in input levels this output response is measured by the marginal product of the input in question (abbreviated MP or MPP for marginal physical product). In discrete terms the marginal product of the ith input is given as

$$MP_i \;=\; \frac{\Delta y}{\Delta x_i} \;=\; \frac{y^2 - y^1}{x_i^2 - x_i^1} \tag{17}$$

where y^2 and x^2 are the level of output and input after the change in the input level and y^1 and x^1 are the levels before the change in input use. For small changes

in x_i the marginal physical product is given by the partial derivative of f(x) with respect to x_i, i.e.,

$$MP_i = \frac{\partial f(x)}{\partial x_i} = \frac{\partial y}{\partial x_i} \tag{18}$$

This is the incremental change in f(x) as x_i is changed holding all other inputs levels fixed. Values of the discrete marginal product for the production function in equation 19 are contained in table 2.7.1.

$$y = 10\,x + 20\,x^2 - 0.60\,x^3 \tag{19}$$

For example the marginal product in going from 4 units of input to 5 units is given by

$$MP_i = \frac{\Delta y}{\Delta x_i} = \frac{475 - 321.6}{5 - 4} = 153.40$$

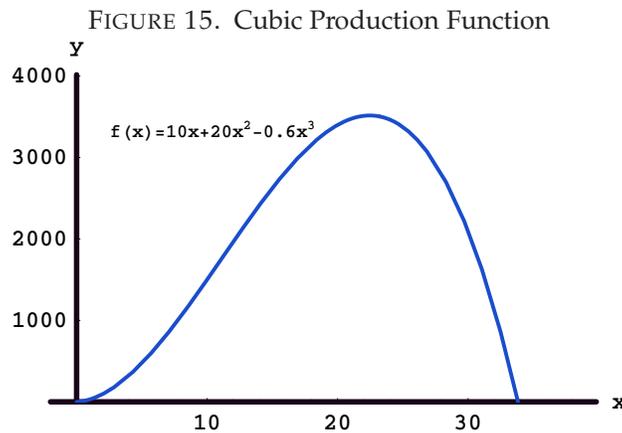The production function in equation 19 is shown in figure 15.

FIGURE 15. Cubic Production Function

TABLE 1. Tabular representation of $y = 10\,x + 20\,x^2 - 0.60\,x^3$

Input (x)	Output (y)	Average Product y/x	Discrete Marginal Product	$\frac{\Delta y}{\Delta x}$ Marginal Product
0.00	0.00			10.00
1.00	29.40	29.40	29.40	48.20
2.00	95.20	47.60	65.80	82.80
3.00	193.80	64.60	98.60	113.80
4.00	321.60	80.40	127.80	141.20
5.00	475.00	95.00	153.40	165.00
6.00	650.40	108.40	175.40	185.20
7.00	844.20	120.60	193.80	201.80
8.00	1052.80	131.60	208.60	214.80
9.00	1272.60	141.40	219.80	224.20
10.0	1500.00	150.00	227.4	230.0
11.0	1731.40	157.40	231.4	232.2
12.0	1963.20	163.60	231.8	230.8
13.0	2191.80	168.60	228.6	225.8
14.0	2413.60	172.40	221.8	217.2
15.0	2625.00	175.00	211.4	205.0
16.0	2822.40	176.40	197.4	189.2
17.0	3002.20	176.60	179.8	169.8
18.0	3160.80	175.60	158.6	146.8
19.0	3294.60	173.40	133.8	120.2
20.0	3400.00	170.00	105.4	90.0

We can compute the marginal product of the production function given in equation 19 using the derivative as follows

$$\frac{dy}{dx} = 10 + 40\,x - 1.80\,x^2 \tag{20}$$

At x = 4 this gives 141.2 while at x = 5 this gives 165.0. The marginal product function for the production function in equation 19 is shown in figure 16.

Notice that it rises at first and then falls as the production function's rate of increase falls. Although we typically do not show the production function and marginal product in the same diagram (because of differences in scale of the vertical axis), figure 17 shows both measures in the same picture to help visualize the relationships between the production function and marginal product.

2.7.2. *Average product (AP).* The marginal product measures productivity of the ith input at a given point on the production function. An average measure of the relationship between outputs and inputs is given by the average product which is just the level of output divided by the level of one of the inputs. Specifically the average product of the ith input is
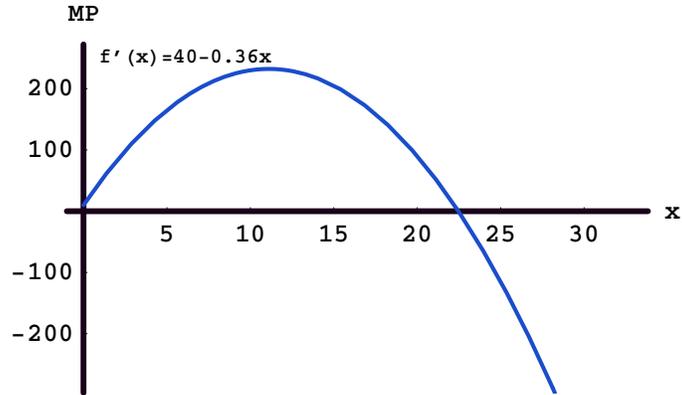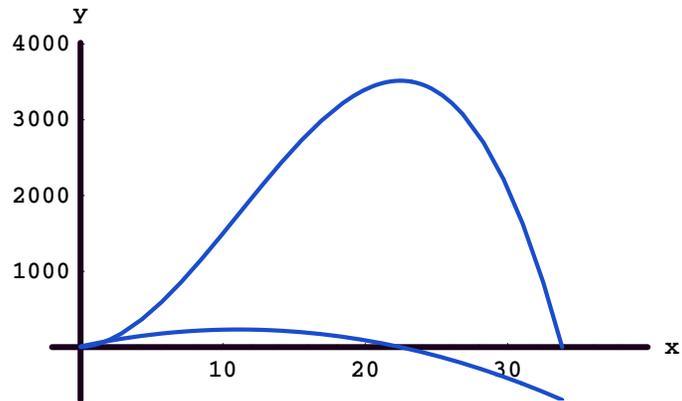
FIGURE 16. Marginal Product
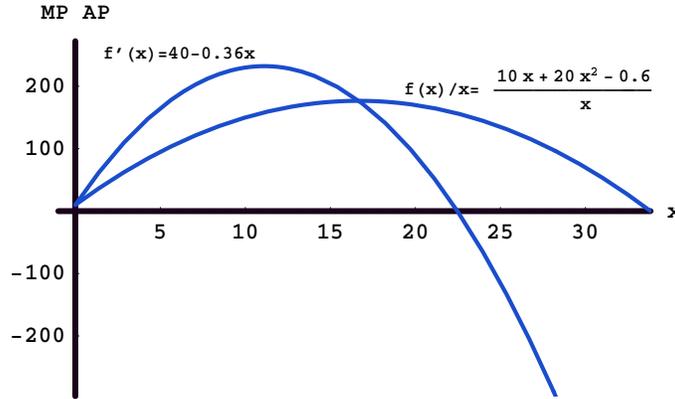

FIGURE 17. Production and Marginal Product

$$AP_i = \frac{f(x)}{x_i} = \frac{y}{x_i} \tag{21}$$

For the production function in equation 19 the average product at x=5 is 475/5 = 95. Figure 18 shows the average and marginal products for the production function in equation 19. Notice that the marginal product curve is above the average product curve when the average product curve is rising. The two curves intersect where the average product reaches its maximum.

We can show that MP = AP at the maximum point of AP by taking the derivative of AP_i with respect to x_i as follows.

FIGURE 18. Average and Marginal Product

MP AP

f'(x)=40-0.36x

f(x)/x= $\dfrac{10\,x + 20\,x^2 - 0.6}{x}$

$$\frac{\partial\left(\frac{f(x)}{x_i}\right)}{\partial x_i} = \frac{x_i\,\frac{\partial f}{\partial x_i} - f(x)}{x_i^2}$$

$$= \frac{1}{x_i}\left(\frac{\partial f}{\partial x_i} - \frac{f(x)}{x_i}\right) \tag{22}$$

$$= \frac{1}{x_i}\,(MP_i - AP_i)$$

If we set the last expression in equation 22 equal to zero we obtain

$$MP_i = AP_i \tag{23}$$

We can represent MP and AP on a production function graph as slopes. The slope of a ray from the origin to a point on f(x) measures average product at that point. The slope of a tangent to f(x) at a point measures the marginal product at that point. This is demonstrated in figure 19.

2.7.3. *Elasticity of ouput.* The elasticity of output for a production function is given by

$$\epsilon_i = \frac{\partial f}{\partial x_i}\frac{x_i}{y} \tag{24}$$

3. ECONOMIES OF SCALE

3.1. **Definitions.** Consider the production function given by

$$y = f(x_1, x_2, \dots x_n) = f(x) \tag{25}$$

where y is output and x is the vector of inputs $x_1 \dots x_n$. The rate at which the amount of output, y, increases as all inputs are increased proportionately is called the degree of returns to scale for the production function f(x). The function f is said to exhibit **nonincreasing returns to scale** if for all $x \in R^n_+$, $\lambda \geq 1$, and $\mu \leq 1$,

$$f(\lambda x) \leq \lambda f(x) \text{ and } \mu f(x) \leq f(\mu x) \tag{26}$$

FIGURE 19. Average and Marginal Product as Slopes
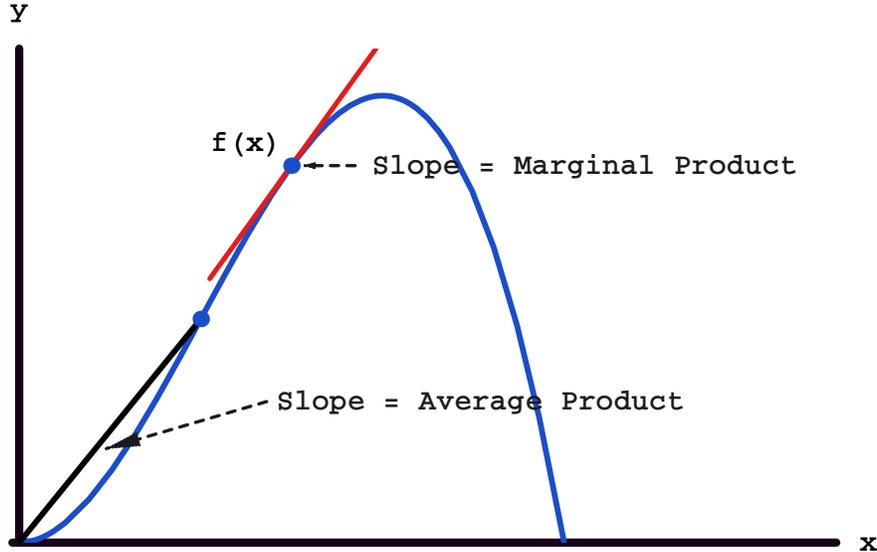
Thus the function increases less than proportionately as all inputs x are increased in the same proportion, and it decreases less than proportionately as all x decrease in the same proportion. When inputs all increase by the same proportion we say that they increase along a ray. In a similar fashion, we say that f exhibits **nondecreasing returns to scale** if for all $x \in R_+^n$, $\lambda \geq 1$, and $0 < \mu \leq 1$.

$$f(\lambda x) \geq \lambda f(x) \text{ and } \mu f(x) \geq f(\mu x) \tag{27}$$

The function f exhibits **constant returns to scale** if for all $x \in R_+^n$ and $\theta > 0$.

$$f(\theta x) = \theta f(x) \tag{28}$$

This global definition of returns to scale is often supplemented by a local one that yields a specific numerical magnitude. This measure of returns to scale will be different depending on the levels of inputs and outputs at the point where it is measured. The elasticity of scale (Ferguson 1971) is implicitly defined by

$$\epsilon = \frac{\partial \ln f(\lambda x)}{\partial \ln \lambda} \mid \lambda = 1 \tag{29}$$

This simply explains how output changes as inputs are changed in fixed proportions (along a ray through the origin). Intuitively, this measures how changes in inputs are scaled into output changes. For one input, the **elasticity of scale is**

$$\epsilon = \frac{\partial f(x)}{\partial x} \frac{x}{f(x)} \tag{30}$$

We can show that the expressions in equations 29 and 30 are the same as follows.

$$\frac{\partial \ln f(\lambda x)}{\partial \ln \lambda} \bigg|_{\lambda=1} = \frac{\partial f(\lambda x)}{\partial \lambda} \frac{\lambda}{f(\lambda x)} \bigg|_{\lambda=1}$$

$$= \left(\frac{\partial f}{\partial \lambda x} x \right) \frac{\lambda}{f(\lambda x)} \bigg|_{\lambda=1}$$

$$= \frac{\partial f}{\partial x} \frac{1}{\lambda} \cdot x \frac{\lambda}{f(\lambda x)} \bigg|_{\lambda=1} \tag{31}$$

$$= \frac{\partial f}{\partial x} \cdot \frac{x}{f(\lambda x)} \bigg|_{\lambda=1}$$

$$= \frac{\partial f}{\partial x} \frac{x}{f(x)}$$

So the elasticity of scale is simply the elasticity of the marginal product of x, i.e.

$$\epsilon = \frac{\partial f(x)}{\partial x} \frac{x}{f(x)} = \frac{\partial y}{\partial x} \frac{x}{y} = \frac{\partial \ln y}{\partial \ln x} \tag{32}$$

In the case of multiple inputs, the elasticity of scale can also be represented as

$$\epsilon = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{x_i}{y}$$

$$= \sum_{i=1}^{n} \frac{\frac{\partial f}{\partial x_i}}{\frac{y}{x_i}} \tag{33}$$

$$= \sum_{i=1}^{n} \frac{MP_i}{AP_i}$$

This can be shown as follows where x is now an n element vector:

$$\frac{\partial \ln f(\lambda x)}{\partial \ln \lambda} \bigg|_{\lambda=1} = \frac{\partial f(\lambda x)}{\partial \lambda} \frac{\lambda}{f(\lambda x)}$$

$$= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial \lambda x_i} x_i \right) \frac{\lambda}{f(\lambda x)} \bigg|_{\lambda=1}$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{1}{\lambda} \cdot x_i \frac{\lambda}{f(\lambda x)} \bigg|_{\lambda=1}$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{x_i}{f(\lambda x)} \bigg|_{\lambda=1} \tag{34}$$

$$= \sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i} \frac{x_i}{f(x)}$$

$$= \sum_{i=1}^{n} MP_i \frac{x_i}{y}$$

Thus elasticity of scale is the sum of the output elasticities for each input. If ϵ is less than one, then the technology is said to exhibit decreasing returns to scale and isoquants spread out as output rises; if it is equal to one, then the technology exhibits constant returns to scale and isoquants are evenly spaced; and if ϵ is greater than one, the technology exhibits increasing returns to scale and the isoquants bunch as output expands. The returns to scale from increasing all of the inputs is thus the average marginal increase in output from all inputs, where each input is weighted by the relative size of that input compared to output. With decreasing returns to scale, the last expression in equation 33 implies that $\mathrm{MP}_i < \mathrm{AP}_i$ for all i.

3.2. **Implications of Various Types of Returns to Scale.** If a technology exhibits constant returns to scale then the firm can expand operations proportionately. If the firm can produce 5 units of output with a profit per unit of $20, then by doubling the inputs and producing 10 units the firm will have a profit of $40. Thus the firm can always make more profits by expanding. If the firm has increasing returns to scale, then by doubling inputs it will have more than double the output. Thus if it makes $20 with 5 units it will make more than $40 with 10 units etc. This assumes in all cases that the firm is increasing inputs in a proportional manner. If the firm can reduce the cost of an increased output by increasing inputs in a manner that is not proportional to the original inputs, then its increased economic returns may be larger than that implied by its scale coefficient.

3.3. **Multiproduct Returns to Scale.** Most firms do not produce a single product, but rather, a number of related products. For example it is common for farms to produce two or more crops, such as corn and soybeans, barley and alfalfa hay, wheat and dry beans, etc. A flour miller may produce several types of flour and a retailer such as Walmart carries a large number of products. A firm that produces several different products is called a multiproduct firm. Consider the production possibility set of the multi-product firm

$$\mathrm{T} \ = \ \{(\mathrm{x, y}) : \ \mathrm{x} \in \mathrm{R}_+^\mathrm{n}, \ \mathrm{y} \in \mathrm{R}_+^\mathrm{m} : \ \mathrm{x \ can \ produce \ y}\}$$

where y and x are vectors of outputs and inputs, respectively. We define the multiproduct elasticity of scale by

$$\epsilon_m \ = \ \sup\{r : \ there \ exists \ a \ \delta > 1 \ such \ that \ (\lambda\, x, \ \lambda\, y^r) \ \in \ T \ for \ 1 \leq \lambda \leq \delta\} \quad (35)$$

For our purposes we can regard the sup as a maximum. The constant of proportion is greater than or equal to 1. This gives the maximum proportional growth rate of outputs along a ray, as all inputs are expanded proportionally [2]. The idea is that we expand inputs by some proportion and see how much outputs can proportionately expand and still be in the production set. If r = 1, then we have constant returns to scale. If r < 1 then we have decreasing returns, and if r > 1, we have increasing returns to scale.

4. RATE OF TECHNICAL SUBSTITUTION

The rate of technical substitution (RTS) measures the extent to which one input substitutes for another input, holding all other inputs constant. The rate of technical substitution is also called the marginal technical rate of substitution or just the marginal rate of substitution.

4.1. **Definition of RTS.** Consider a production function given by

$$y = f(x_1, x_2 \ldots, x_n) \tag{36}$$

If the implicit function theorem holds then

$$\phi(y, x_1, x_2, \ldots, x_n) = y - f(x_1, x_2 \ldots, x_n) = 0 \tag{37}$$

is continuously differentiable and the Jacobian matrix has rank 1. i.e.,

$$\frac{\partial \phi}{\partial x_j} = \frac{\partial f}{\partial x_j} \neq 0 \tag{38}$$

Given that the implicit function theorem holds, we can solve equation 38 for x_k as a function of y and the other x's i.e.

$$x_k^* = \psi_k(x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, y) \tag{39}$$

Thus it will be true that

$$\phi(y, x_1, x_2, \ldots, x_{k-1}, x_k^*, x_{k+1}, \ldots, x_n) \equiv 0 \tag{40}$$

or that

$$y \equiv f(x_1, x_2, \ldots, x_{k-1}, x_k^*, x_{k+1}, \ldots, x_n) \tag{41}$$

Differentiating the identity in equation 41 with respect to x_j will give

$$0 = \frac{\partial f}{\partial x_j} + \frac{\partial f}{\partial x_k} \frac{\partial x_k}{\partial x_j} \tag{42}$$

or

$$\frac{\partial x_k}{\partial x_j} = \frac{-\frac{\partial f}{\partial x_j}}{\frac{\partial f}{\partial x_k}} = RTS = MRS \tag{43}$$

Or we can obtain this directly as

$$\frac{\partial \phi(y, x_1, x_2, \ldots, x_{k-1}, \psi_k, x_{k+1}, \ldots, x_n)}{\partial x_k} \frac{\partial \psi_k}{\partial x_j} = \frac{-\partial \phi(y, x_1, x_2, \ldots, x_{k-1}, \psi_k, x_{k+1}, \ldots, x_n)}{\partial x_j}$$

$$\Rightarrow \frac{\partial \phi}{\partial x_k} \frac{\partial x_k}{\partial x_j} = - \frac{\partial \phi}{\partial x_j}$$

$$\Rightarrow \frac{\partial x_k}{\partial x_j} = \frac{-\frac{\partial f}{\partial x_j}}{\frac{\partial f}{\partial x_k}} = MRS$$

$$\tag{44}$$

The above expression represents the slope of the projection of the boundary of the input requirement (or level/contour) set V(y) into x_k, x_j space. With two inputs, this is, of course, just the slope of the boundary. Its slope is negative. It is

convex because V(y) is convex. Because it is convex, there will be a diminishing rate of technical substitution. Figure 20 shows the rate of technical substitution.

FIGURE 20. Rate of Technical Substitution
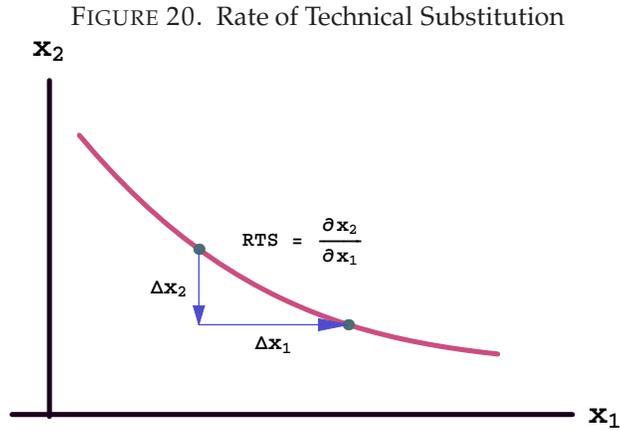
4.2. **Example Computation of RTS.**

4.2.1. *Cobb-Douglas.* Consider the following Cobb-Douglas production function

$$y = 5\, x_1^{\frac{1}{3}} x_2^{\frac{1}{4}} \tag{45}$$

The partial derivative of y with respect to x_1 is

$$\frac{\partial y}{\partial x_1} = \frac{5}{3}\, x_1^{\frac{-2}{3}} x_2^{\frac{1}{4}} \tag{46}$$

The partial derivative of y with respect to x_2 is

$$\frac{\partial y}{\partial x_2} = \frac{5}{4}\, x_1^{\frac{1}{3}} x_2^{\frac{-3}{4}} \tag{47}$$

The rate of technical substitution is

$$
\begin{aligned}
\frac{\partial x_2}{\partial x_1} &= \frac{-\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}} = -\frac{\frac{5}{3} x_1^{\frac{-2}{3}} x_2^{\frac{1}{4}}}{\frac{5}{4} x_1^{\frac{1}{3}} x_2^{\frac{-3}{4}}} \\
&= -\frac{\frac{5}{3}}{\frac{5}{4}} \frac{x_2}{x_1} \\
&= -\frac{4}{3} \frac{x_2}{x_1}
\end{aligned}
\tag{48}
$$

4.2.2. *CES.* Consider the following CES production function

$$y = 5 \left[0.6 \, x_1^{-2} + 0.2 \, x_2^{-2} \right]^{\frac{-1}{2}} \tag{49}$$

The partial derivative of y with respect to x_1 is

$$\frac{\partial y}{\partial x_1} = \frac{-5}{2} \left[0.6 \, x_1^{-2} + 0.2 \, x_2^{-2} \right]^{\frac{-3}{2}} (-6/5) \, x_1^{-3} \tag{50}$$

The partial derivative of y with respect to x_2 is

$$\frac{\partial y}{\partial x_2} = \frac{-5}{2} \left[0.6 \, x_1^{-2} + 0.2 \, x_2^{-2} \right]^{\frac{-3}{2}} (-2/5) \, x_2^{-3} \tag{51}$$

The rate of technical substitution is

$$\begin{aligned}
\frac{\partial x_2}{\partial x_1} &= -\frac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}} = -\frac{\frac{-5}{2} \left[0.6 \, x_1^{-2} + 0.2 \, x_2^{-2} \right]^{\frac{-3}{2}} (-6/5) \, x_1^{-3}}{\frac{-5}{2} \left[0.6 \, x_1^{-2} + 0.2 \, x_2^{-2} \right]^{\frac{-3}{2}} (-2/5) \, x_2^{-3}} \\
&= -\frac{\frac{2}{5}}{\frac{6}{5}} \frac{x_1^{-3}}{x_2^{-3}} \\
&= -3 \frac{x_2^3}{x_1^3}
\end{aligned} \tag{52}$$

5. ELASTICITY OF SUBSTITUTION

The elasticity of substitution is a unitless measure of how various inputs substitute for each other. For example, how does capital substitute for labor, how does low skilled labor substitute for high skilled labor, how do pesticides substitute for tillage, how does ethanol substitute for gasoline. The elasticity of substituion attempts to measure the curvature of the lower boundary of the input requirement set. The most commonly used measure of the elasticity of substitution based on the slope of an isoquant is due to Hicks [13, p. 117, 244-245], [17, p.330]. He defines the elasticity of substitution between x_2 and x_1 as follows.

$$\sigma = \frac{d(x_2/x_1)}{d(f_1/f_2)} \cdot \frac{(f_1/f_2)}{(x_2/x_1)} \tag{53}$$

This is the percentage change in the input ratio induced by a one percent change in the RTS.

5.1. **Geometric Intuition.** We can better understand this definition by an appeal to geometry. Consider figure 21. First consider the factor ratio of x_2 to x_1. Along the ray labeled *a*, the ratio $\frac{x_2}{x_1}$ is given by the tangent of the angle θ. Along the ray labeled *b*, the ratio $\frac{x_2}{x_1}$ is given by the tangent of the angle ϕ. For example at point d, $\frac{x_2^0}{x_1^0} = \tan \phi$ along the ray *b*.

Now consider the ratio of the slopes of the input requirement boundary at two different points. Figures 22 and 23 show these slopes. The slope of the curve at point *c* is equal to minus the tangent of the angle γ in figure 22. The slope of the curve at point *d* is equal to minus the tangent of the angle δ in figure 23.

FIGURE 21. Elasticity of Substitution (Factor Ratios)
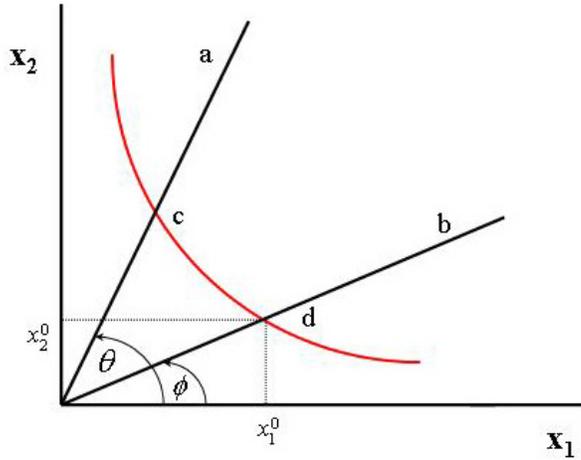
Consider the right triangle formed by drawing a vertical line from the intersection of the two tangent lines and the x_1 axis. The base is the x_1 axis and the hypotenuse is the tangent line between the other two sides. This is shown in figure 24. Angle α measures the third angle in the smaller triangle. Angle α + angle γ equal 90 degrees. In figure 25, the angle between the vertical line and the tangent at point d is represented by $\alpha + \beta$. Angle β + angle α+ angle δ also equals 90 degrees, so that $\alpha + \gamma = \alpha + \beta + \delta$ or $\gamma = \beta + \delta$. This then means that β, the angle between the two tangent lines is equal to γ - δ.

FIGURE 22. Elasticity of Substitution (Angle of Tangent at Point c)
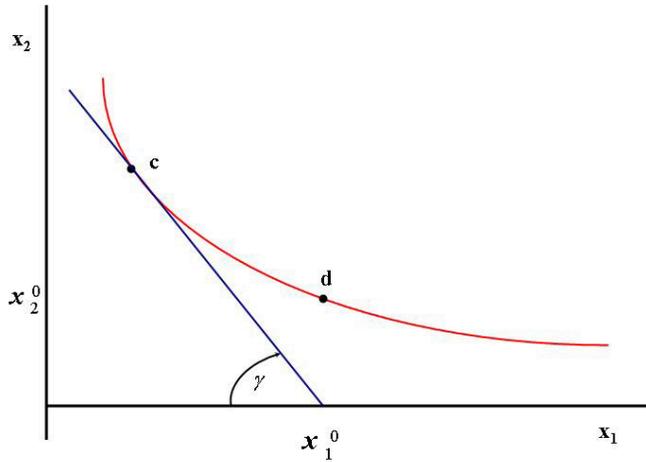
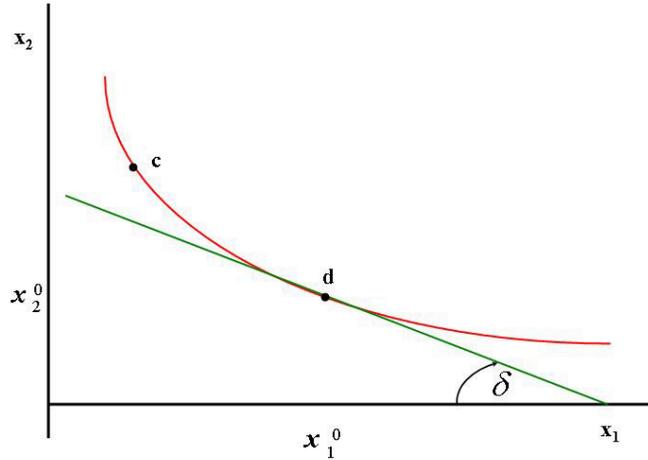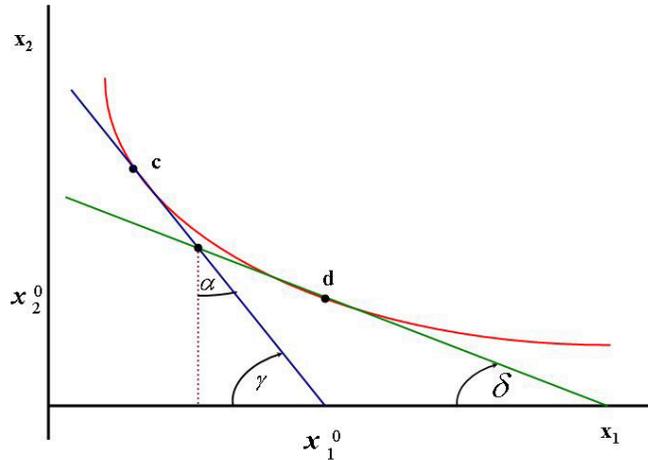FIGURE 23. Elasticity of Substitution (Angle of Tangent at Point d)

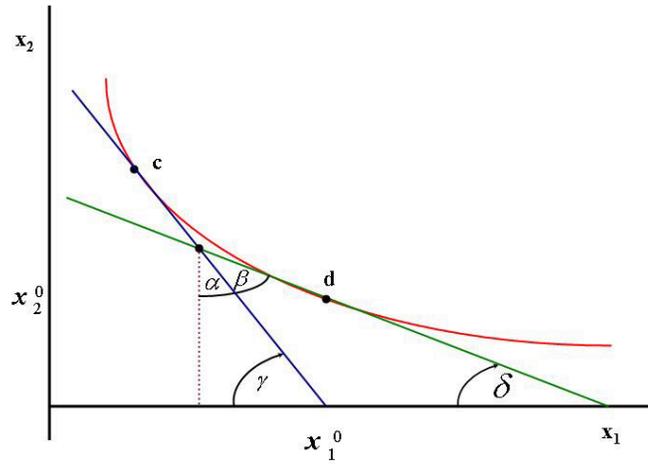FIGURE 24. Elasticity of Substitution (Angle Tangent at Point d)

If we combine the information in figures 22, 23, 24 and 25 into figure 26, we can measure the elasticity of substitution. Remember that it is given by

$$\sigma \;=\; \frac{d(x_2/x_1)}{d(f_1/f_2)} \;\cdot\; \frac{(f_1/f_2)}{(x_2/x_1)}$$

The change in $\frac{x_2}{x_1}$ is given by the angle ξ which is θ - ϕ. The marginal rate of technical substitution is given by the slope of the boundary of V(y). At point d the slope is given by δ while at c it is given by γ. The change in this slope is β. The rate of technical substitution is computed as

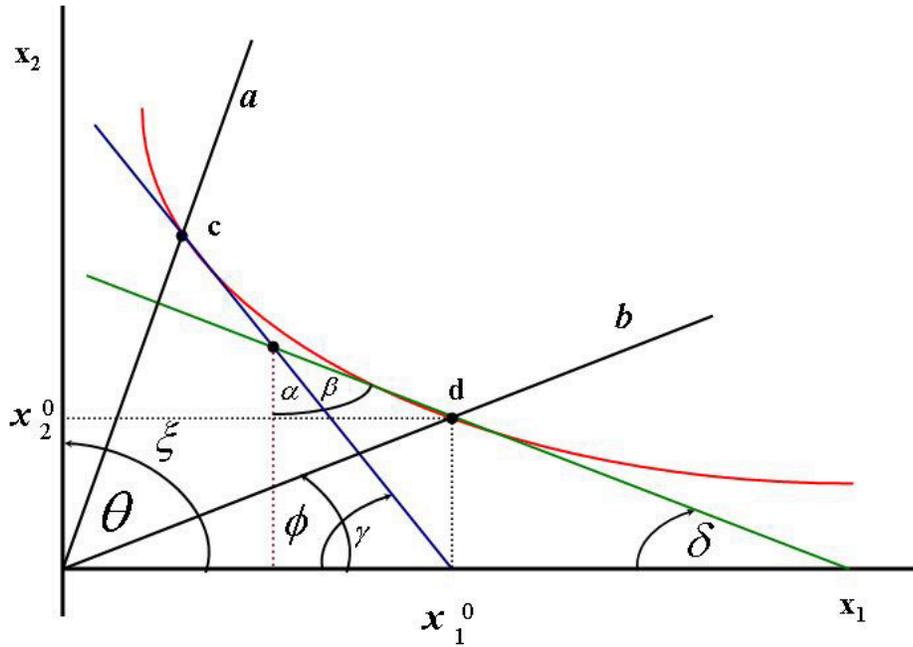FIGURE 25. Elasticity of Substitution Measuring Slope Change

$$RTS = \frac{\partial x_2}{\partial x_1} = \frac{-\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}}$$

Therefore we can say that the ratio of the change in the input ratio and the change in the slope of the lower the boundary of V(y) is given by

$$\sigma \approx \frac{\xi}{\beta}$$

FIGURE 26. Geometric Interpretation of the Elasticity of Substitution

5.2. **Algebraic Expression for σ_{12}.** In the case of two inputs, the elasticity of substitution is given by

$$\sigma_{12} = \frac{-f_1 \ f_2 \ (x_1 \ f_1 \ + \ x_2 \ f_2)}{x_1 \ x_2 \ (f_{11} \ f_2^2 \ - \ 2f_{12} \ f_1 \ f_2 \ + \ f_{22} \ f_1^2)} \tag{54}$$

5.3. **Derivation of the Algebraic Expression for σ_{12}.** By definition

$$\sigma_{12} = \frac{d(x_2/x_1)}{d(f_1/f_2)} \cdot \frac{(f_1/f_2)}{(x_2/x_1)} \tag{55}$$

First compute $d\left(\frac{x_2}{x_1}\right)$. This is a differential. Computing this differential we obtain

$$d\left(\frac{x_2}{x_1}\right) = \frac{x_1 \ dx_2 \ - \ x_2 \ dx_1}{x_1^2} \tag{56}$$

Then compute the differential $d\left(\frac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}}\right)$.

$$d\left(\frac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}}\right) = d\left(\frac{f_1}{f_2}\right) = \frac{\partial \left(\frac{f_1}{f_2}\right)}{\partial x_1} \ dx_1 \ + \ \frac{\partial \left(\frac{f_1}{f_2}\right)}{\partial x_2} \ dx_2 \tag{57}$$

With the level of y fixed as along the boundary of V(y), we have via the implicit function theorem or equation 43

$$\frac{\partial x_2}{\partial x_1} = \frac{-\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}} \tag{58}$$

Rearranging equation 58 (and being a bit sloppy with differentials and partial derivatives) we can conclude

$$\frac{dx_2}{dx_1} = \frac{-f_1}{f_2}$$

$$\Rightarrow dx_2 = \left(\frac{-f_1}{f_2}\right) dx_1 \tag{59}$$

and

$$dx_1 = \left(\frac{-f_2}{f_1}\right) dx_2 \tag{60}$$

This then implies that

$$d\left(\frac{x_2}{x_1}\right) = \frac{x_1 \left(\frac{-f_1}{f_2}\right) dx_1 - x_2 dx_1}{x_1^2}$$

$$= \frac{\left(x_1 \left(\frac{-f_1}{f_2}\right) - x_2\right) dx_1}{x_1^2} \tag{61}$$

and that

$$d\left(\frac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}}\right) = d\left(\frac{f_1}{f_2}\right) = \frac{\partial \left(\frac{f_1}{f_2}\right)}{\partial x_1} dx_1 + \frac{\partial \left(\frac{f_1}{f_2}\right)}{\partial x_2} \left(\frac{-f_1}{f_2}\right) dx_1$$

$$= \left(\frac{\partial \left(\frac{f_1}{f_2}\right)}{\partial x_1} + \frac{\partial \left(\frac{f_1}{f_2}\right)}{\partial x_2} \left(\frac{-f_1}{f_2}\right)\right) dx_1 \tag{62}$$

Now compute $\frac{\partial \left(\frac{f_1}{f_2}\right)}{\partial x_2}$.

$$\frac{\partial \left(\frac{f_1}{f_2}\right)}{\partial x_2} = \frac{f_2 f_{12} - f_1 f_{22}}{f_2^2} \tag{63}$$

and $\frac{\partial \left(\frac{f_1}{f_2}\right)}{\partial x_1}$

$$\frac{\partial \left(\frac{f_1}{f_2}\right)}{\partial x_1} = \frac{f_2 f_{11} - f_1 f_{21}}{f_2^2} \tag{64}$$

Now replace $\frac{\partial \left(\frac{f_1}{f_2}\right)}{\partial x_2}$ and $\frac{\partial \left(\frac{f_1}{f_2}\right)}{\partial x_1}$ in equation 62 with their equivalent expressions from equations 63 and 64 in equation 65.

$$d\left(\frac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}}\right) = d\left(\frac{f_1}{f_2}\right) = \left(\frac{f_2 f_{11} - f_1 f_{21}}{f_2^2} + \frac{f_2 f_{12} - f_1 f_{22}}{f_2^2} \left(\frac{-f_1}{f_2}\right)\right) dx_1 \tag{65}$$

We now have all the pieces we need to compute σ_{12} by substituing from equations 61 and 65 into equation 55.

$$
\begin{aligned}
\sigma_{12} &= \frac{d(x_2/x_1)}{d(f_1/f_2)} \cdot \frac{(f_1/f_2)}{(x_2/x_1)} \\[2mm]
&= \frac{\dfrac{\left(x_1\left(\frac{-f_1}{f_2}\right) - x_2\right) dx_1}{x_1^2}}{\left(\dfrac{f_2\,f_{11} - f_1\,f_{21}}{f_2^2} + \dfrac{f_2\,f_{12} - f_1\,f_{22}}{f_2^2}\left(\frac{-f_1}{f_2}\right)\right) dx_1} \cdot \frac{(f_1/f_2)}{(x_2/x_1)} \\[2mm]
&= \frac{\left(x_1\left(\frac{-f_1}{f_2}\right) - x_2\right) f_1\,x_1}{x_1^2\,f_2\,x_2\left(\dfrac{f_2\,f_{11} - f_1\,f_{21}}{f_2^2} + \dfrac{f_2\,f_{12} - f_1\,f_{22}}{f_2^2}\left(\frac{-f_1}{f_2}\right)\right)} \\[2mm]
&= \frac{\dfrac{(-f_1\,x_1 - f_2\,x_2)}{f_2}\,f_1}{x_1\,x_2\,f_2\left(\dfrac{f_2\,f_{11} - f_1\,f_{21}}{f_2^2} + \dfrac{f_2\,f_{12} - f_1\,f_{22}}{f_2^2}\left(\frac{-f_1}{f_2}\right)\right)} \\[2mm]
&= \frac{-f_1\,(f_1\,x_1 + f_2\,x_2)}{x_1\,x_2\,f_2^2\left(\dfrac{f_2\,f_{11} - f_1\,f_{21}}{f_2^2} + \dfrac{f_2\,f_{12} - f_1\,f_{22}}{f_2^2}\left(\frac{-f_1}{f_2}\right)\right)}
\end{aligned}
\tag{66}
$$

Consider the expression in parentheses in the denominator of equation 66. We can rearrange and simplify it as follows

$$
\begin{aligned}
\left(\frac{f_2 f_{11} - f_1 f_{21}}{f_2^2} + \frac{f_2\,f_{12} - f_1 f_{22}}{f_2^2}\left(\frac{-f_1}{f_2}\right)\right) &= \frac{f_2^2 f_{11} - f_1 f_2 f_{21}}{f_2^3} + \frac{-f_1 f_2 f_{12} + f_1^2 f_{22}}{f_2^3} \\[2mm]
&= \frac{f_2^2 f_{11} - f_1 f_2 f_{21} - f_1 f_2 f_{12} + f_1^2 f_{22}}{f_2^3} \\[2mm]
&= \frac{f_2^2 f_{11} - 2 f_1 f_2 f_{12} + f_1^2 f_{22}}{f_2^3} \quad \text{by Young's Theorem}
\end{aligned}
\tag{67}
$$

Now substitute equation 67 into equation 66

$$
\begin{aligned}
\sigma_{12} &= \frac{-f_1\,(f_1\,x_1 + f_2\,x_2)}{x_1\,x_2\,f_2^2\left(\dfrac{f_2^2 f_{11} - 2 f_1 f_2 f_{12} + f_1^2 f_{22}}{f_2^3}\right)} \\[2mm]
&= \frac{-f_1\,f_2\,(f_1\,x_1 + f_2\,x_2)}{x_1\,x_2\,f_2^2\,(f_{11}\,f_2^2 - 2\,f_{12}\,f_1\,f_2 + f_{22}\,f_1^2)}
\end{aligned}
\tag{68}
$$

5.4. **Matrix Representation of σ_{12}.** In matrix notation σ_{12} is given by

$$
\sigma_{12} = \frac{(x_1\,f_1 + f_2\,x_2)}{x_1\,x_2}\,\frac{F_{12}}{F}
\tag{69}
$$

where

$$
F = \begin{vmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{vmatrix} = \det(H_B)
\tag{70}
$$

where H_B is the bordered Hessian of the production function. F_{12} is the cofactor of f_{12} in the matrix F, i.e.,

$$f_{12} = (-1)^5 \begin{vmatrix} 0 & f_1 \\ f_2 & f_{21} \end{vmatrix} = (-1)(-f_1 \, f_2) = f_1 \, f_2 \tag{71}$$

The determinant of F computed by expanding along the first row is given by

$$F = \begin{vmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{vmatrix} = 0 \cdot (f_{11}f_{22} - f_{12}^2) - f_1 \cdot (f_1 f_{22} - f_2 f_{12}) + f_2 \cdot (f_1 \, f_{21} - f_2 f_{11})$$

$$= -f_1^2 \, f_{22} + 2 f_1 \, f_2 \, f_{12} - f_2^2 \, f_{11}$$
$$\tag{72}$$

Substitute equations 71 and 70 into the last expression of equation 69 to find $\frac{F_{12}}{F}$

$$\frac{F_{12}}{F} = \frac{-f_1 \, f_2}{f_1^2 \, f_{22} - 2 f_1 \, f_2 \, f_{12} + f_2^2 \, f_{11}} \tag{73}$$

Now substitute $\frac{F_{12}}{F}$ from equation 73 into equation 69.

$$\sigma_{12} = \frac{(x_1 \, f_1 + f_2 \, x_2)}{x_1 \, x_2} \frac{-f_1 \, f_2}{f_1^2 \, f_{22} - 2 f_1 \, f_2 \, f_{12} + f_2^2 \, f_{11}} \tag{74}$$

Then compare with equation 68

$$\sigma_{12} = \frac{-f_1 \, f_2 \, (f_1 \, x_1 + f_2 \, x_2)}{x_1 \, x_2 \, f_2^2 \, (f_{11} \, f_2^2 - 2 \, f_{12} \, f_1 \, f_2 + f_{22} \, f_1^2)}$$

Note that from equation 74, σ_{12} is symmetric. We can also show that for a quasi-concave production function that $\frac{F_{12}}{F}$ is always negative so that the elasticity of substitution is also negative. This also follows from the fact that quasi-concave functions have convex level sets.

6. Homgeneity

6.1. Definition of Homogeneity. A function is homogeneous of degree k in x if

$$f(\lambda x, z) = \lambda^k f(x, z), \forall \lambda > 0, \forall x \in R_+^n, \forall z \in R_+^m \tag{75}$$

6.2. Some Properties of Homogeneous Functions. If $f(x_1, \ldots, x_n)$ is homogeneous of degree k in x, then with suitable restrictions on the function f in each case,

a:
$$\frac{\partial f}{\partial x_i} \text{ is homogeneous of degree k-1, i} = 1, 2, \ldots, n \tag{76}$$

b:
$$f(x) = x_i^k f\left(\frac{x_1}{x_i}, \frac{x_2}{x_i}, \ldots, 1, x_{\frac{i+1}{x_i}}, \ldots, \frac{x_n}{x_i}\right),$$
$$x_i > 0, \ i = 1, 2, \ldots, n. \tag{77}$$

6.3. **An Euler Equation for Homogeneous Functions.** f is homogeneous of degree k if and only if

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \, x_i \;=\; k \, f(x) \tag{78}$$

Proof of Euler equation. Define the following functions g(λ, x) and h(x).

$$g(\lambda, x) \;=\; f(\lambda \, x) \;-\; \lambda^k \, f(x) \;\equiv\; 0 \tag{79a}$$

$$h(x) \;=\; \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \, x_i \;-\; k \, f(x) \tag{79b}$$

Now differentiate g with respect to λ using the chain rule

$$\begin{aligned}
\frac{\partial g(\lambda, x)}{\partial \lambda} &= \sum_{i=1}^{n} \frac{\partial f(\lambda x)}{\partial (\lambda x_i)} \frac{\partial (\lambda x_i)}{\partial \lambda} \;-\; k\lambda^{k-1} \, f(x) \\
&= \sum_{i=1}^{n} \frac{\partial f(\lambda x)}{\partial (\lambda x_i)} \, x_i \;-\; k\lambda^{k-1} \, f(x)
\end{aligned} \tag{80}$$

Now multiply both sides of 80 by λ to obtain

$$\begin{aligned}
\lambda \frac{\partial g(\lambda, x)}{\partial \lambda} &= \lambda \left[\Sigma_{i=1}^{n} \frac{\partial f(\lambda x)}{\partial \lambda x_i} \, x_i \;-\; k\lambda^{k-1} \, f(x) \right] \\
&= \left[\Sigma_{i=1}^{n} \frac{\partial f(\lambda x)}{\partial \lambda x_i} \, \lambda x_i \;-\; k\lambda^{k} \, f(x) \right] \\
&= \left[\Sigma_{i=1}^{n} \frac{\partial f(\lambda x)}{\partial \lambda x_i} \, \lambda x_i \;-\; k \, f(\lambda \, x) \right] \\
&= h(\lambda x)
\end{aligned} \tag{81}$$

The last two steps follows because f(λx)= λ^k f(x). By assumption g(λ, x) $\equiv$ 0 so $\frac{\partial g(\lambda, x)}{\partial \lambda} \equiv 0$. Now set $\lambda = 1$ in the equation 81 obtain

$$\begin{aligned}
0 \;=\; h(\lambda x)|_{\lambda=1} &= \left[\Sigma_{i=1}^{n} \frac{\partial f(x)}{\partial x_i} \, x_i \;-\; k \, f(x) \right] \\
&\Rightarrow \Sigma_{i=1}^{n} \frac{\partial f(x)}{\partial x_i} \, x_i \;=\; k \, f(x)
\end{aligned} \tag{82}$$

$\square$

A more complete disussion of homogeneity is contained in Eichorn [6, p. 69-77],Sydsaeter [20, p. 125-131] and Sydsaeter and Hammond. [21, p. 432-442].

6.4. **Geometry of Homogeneous Functions.** There are a number of ways to graphically characterize a homothetic function. The first is that the points on the graph above a ray form a smooth curve in $n + 1$ space. The second considers the shape of level sets and the third the slope of level sets along a given ray.

FIGURE 27. Graph of a PLH Function
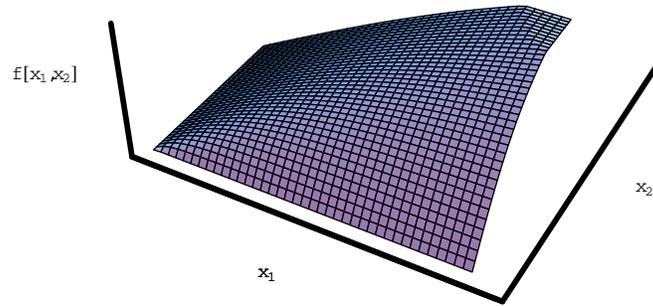
6.4.1. *Graph of a homogeneous function as lines through the origin.* Consider a positive linear homogeneous production (PLH) function with two inputs. The graph of the function is shown in figure 27.

In figure 28 we plot the points on the surface of the graph which lie above the ray running through the point (x10, x20, 0). As can be clearly seen, the graph above this ray is a straight line. The height of the function at this point is c. Points along the ray through the point (x10, x20, 0) are generated as λ (x10, x20, 0). The height of the function above any point along the ray is then given by λ^k c, where k is the degree of homogeneity of the function. Figure 29 shows the straight line as part of the graph.

Thus the graph of f consists of curves of the form z = λ^k c, above each ray in the hyperplane generated by λ (x10,x20,0). Thus if we know one point on each ray we know the function. If k = 1 then z = λ c and the graph is generated by straight lines through the origin. This is the case for the function in this example as demonstated in figure 30. Figures 31, 32, 33, and 34 show the same relationships when the function is homogeneous of degree 0 < k < 1.

6.4.2. *Shape of level sets.* Suppose the $\tilde{x}$ and $\tilde{z}$ are both elements of the lower boundary of a particular level set for a function which is homogeneous of degree k. Then we know that

$$
\begin{aligned}
f(\tilde{x}) &= f(\tilde{z}) \\
f(\lambda\tilde{x}) &= \lambda^k f(\tilde{x}) \\
f(\lambda\tilde{z}) &= \lambda^k f(\tilde{z}) \\
\Rightarrow f(\lambda\tilde{x}) &= f(\lambda\tilde{z})
\end{aligned}
\tag{83}
$$

FIGURE 28. Graph of PLH Function as a Straight Line Above Any Ray

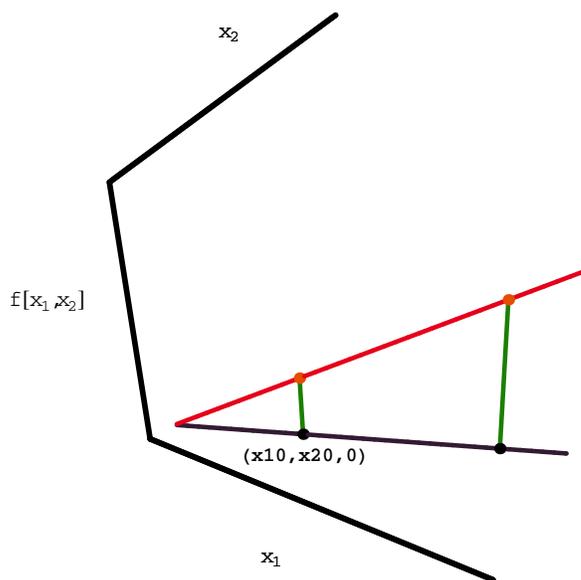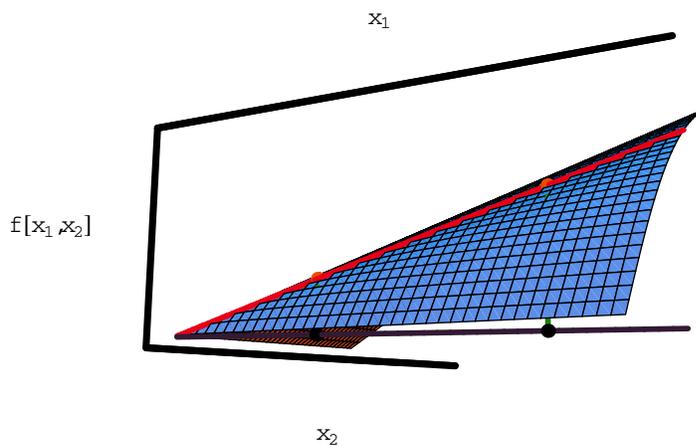

FIGURE 29. Graph of a PLH Function Along with Function Value
Above a Ray

Thus the two points $\lambda\tilde{x}$ and $\lambda\tilde{z}$ are on the same level set. In figure 35 we can see how to construct the level set containing $\lambda\tilde{x}$ and $\lambda\tilde{z}$ and the one containing $\tilde{x}$ and $\tilde{z}$. If we scale points a and b on the intial lower isoquant by 2, we arrive on the same level set at y = 108.15. Given that this is less than twice 60, this technology exhibits decreasing returns to scale. We can construct all other level sets from this initial one in a similar manner. The conclusion is that if the function is homogeneous

FIGURE 30. Graph of a PLH function Along with Function Values Above Rays
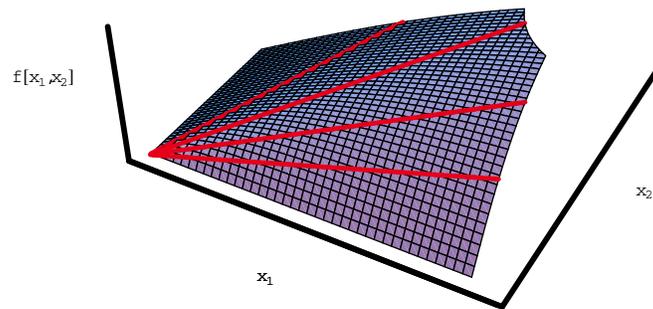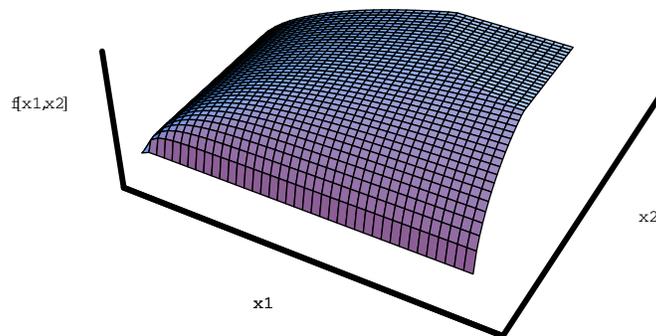

FIGURE 31. Graph of a General Homogeneous Function

and we know one level set, we can construct all the others by a radial expansion or contraction of the set.

6.4.3. *Description using slopes along rays.* Remember from equation 76 that the derivative function homogeneous of degree k is homogenous of degree $k - 1$. Specifically

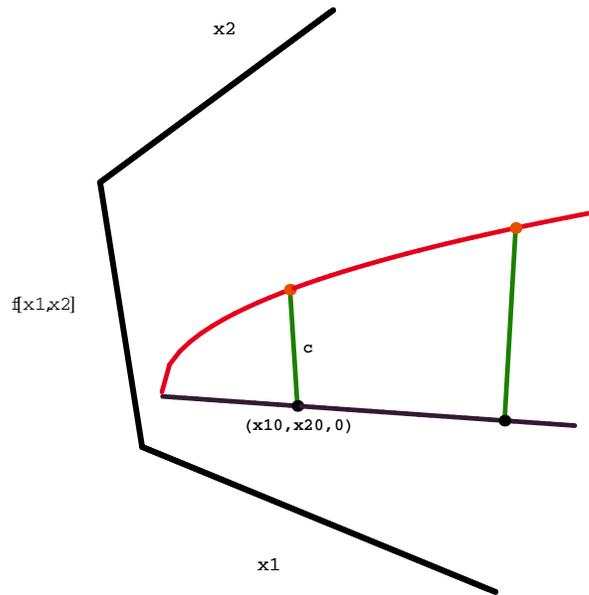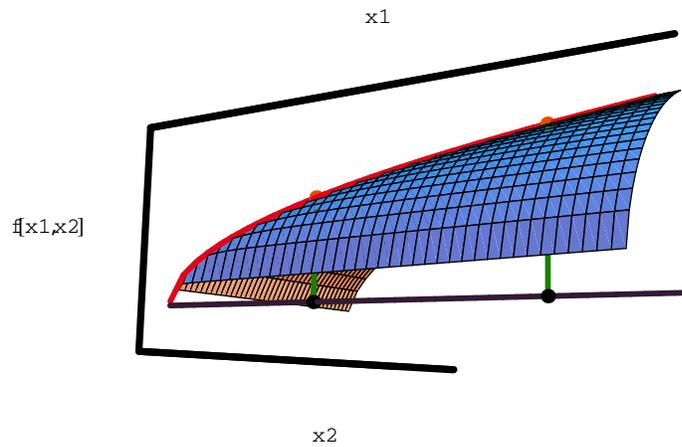FIGURE 32. Graph of General Homogeneous Function as a Curve
Above Any Ray

FIGURE 33. Graph of a general homogeneous function along with
function value above a ray

$$\frac{\partial f}{\partial x_i}\left(\lambda x_1, \lambda x_2, \cdots, \lambda x_n\right) = \lambda^{k-1}\frac{\partial f}{\partial x_i}\left(\lambda x_1, \lambda x_2, \cdots, \lambda x_n\right) \tag{84}$$

This is true for all i. Define the gradient of f(x_1, x_2, ..., x_n) as the vector of first
derivatives, i.e.,

FIGURE 34. Graph of a general homogeneous function along with function values above rays
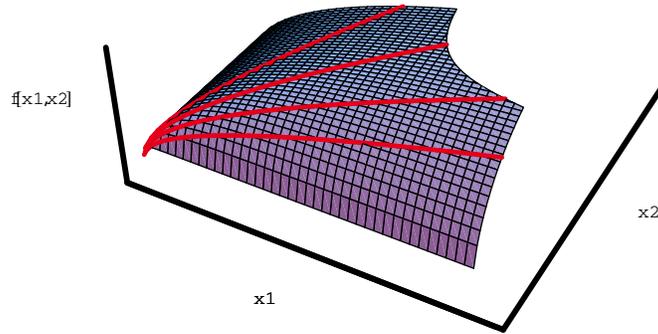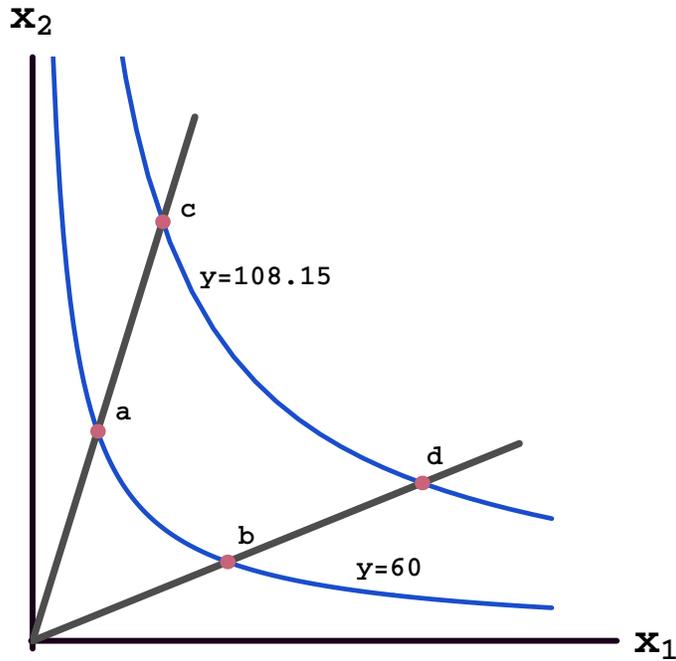

FIGURE 35. Constructing One Level Set from an Initial One for Homogeneous Functions


$$\nabla f(x) \ = \ \begin{pmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{pmatrix} \tag{85}$$

We can then write for all $\lambda > 0$ and i = 1,2,..., n

$$\nabla f(\lambda x_1, \lambda x_2, \cdots, \lambda x_n) \;=\; \lambda^{k-1} \, \nabla f(\lambda x_1, \lambda x_2, \cdots, \lambda x_n) \tag{86}$$

So along a given ray the gradients are parallel. Gradients are orthogonal (or at right angles) to level sets. So along a given ray if the gradients are all parallel, then the level sets are parallel along the ray.

FIGURE 36. Level Sets for a Function which is Homogeneous of Degree One
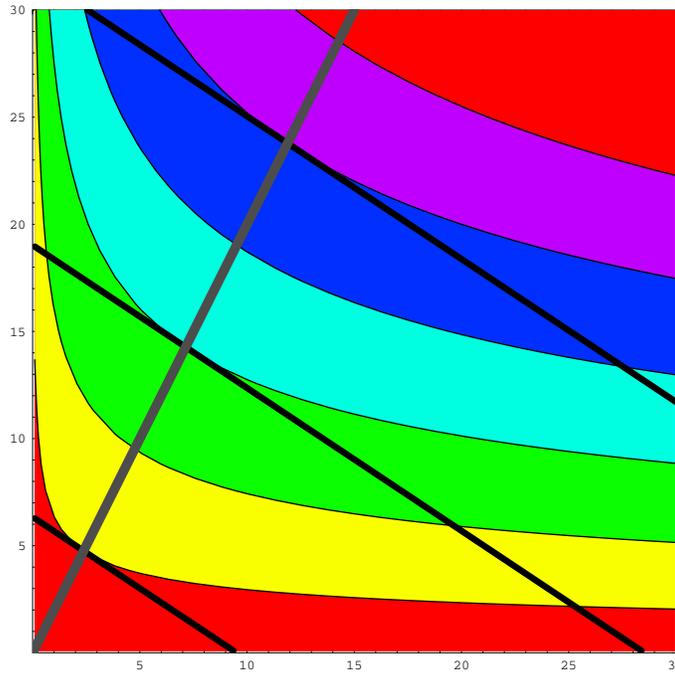
6.5. **Returns to Scale and a Generalized Euler Equation.**

6.5.1. *Definition of elasticity of scale.*

$$\begin{aligned}
\epsilon(\lambda, x) \;&=\; \frac{\partial \ln f(\lambda x)}{\partial \ln \lambda} \;=\; \frac{\partial f(\lambda x)}{\partial \lambda} \, \frac{\lambda}{f(\lambda x)} \\
&= \sum_{i=1}^{n} \frac{\partial f(\lambda x)}{\partial \lambda x_i} \, \frac{\partial(\lambda x_i)}{\partial \lambda} \, \frac{\lambda}{f(\lambda x)} \\
&= \sum_{i=1}^{n} \frac{\partial f(\lambda x)}{\partial \lambda x_i} \, \frac{\lambda x_i}{f(\lambda x)}
\end{aligned} \tag{87}$$

Now evaluate at λ = 1 to obtain

$$\epsilon(1, x) = \sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i} \frac{x_i}{f(x)}$$

$$\Rightarrow \sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i} x_i = \epsilon(1, x) f(x) \tag{88}$$

The sum of the partial derivatives of f(x) multiplied by the levels of the x's is equal to $\epsilon(1,x)$ multiplied by the value of the function f(x). Remember from equation 24 the output elasticity is given by

$$\epsilon_j = \frac{\partial f(x)}{\partial x_j} \frac{x_j}{f(x)}, \quad f(x) \neq 0. \tag{89}$$

Replace 1 in equation 88 with λ and rewrite it as follows

$$\epsilon(\lambda, x) = \sum_{i=1}^{n} \frac{\partial f(\lambda x)}{\partial \lambda x_i} \frac{\lambda x_i}{f(\lambda x)} \tag{90}$$

Now substitute equation 89 in equation 90 to obtain

$$\epsilon(\lambda x) = \epsilon_1(\lambda x) + \epsilon_2(\lambda x) + \cdots + \epsilon_n(\lambda x) \tag{91}$$

If $\lambda = 1$, then we obtain the result we previously obtained in equation 33.

$$\epsilon(1, x) = \epsilon_1(x) + \epsilon_2(x) + \cdots + \epsilon_n(x) \tag{92}$$

And we usually write $\epsilon(x)$ for $\epsilon(1,x)$.

6.5.2. *Properties of returns to scale.* Remember from equations 26, 27, and 28 that

1: $\epsilon(x) = 1 \Rightarrow$ constant returns to scale (CRS)
2: $\epsilon(x) < 1 \Rightarrow$ decreasing returns to scale (DRS)
3: $\epsilon(x) > 1 \Rightarrow$ increasing returns to scale (IRS)

6.5.3. *General nature of returns to scale.* In general $\epsilon(x)$ is a function depending on x. With different levels of x_j, $\epsilon(x)$ will change.

6.6. **Returns to Scale for Homogeneous Technologies.** We know from equation 88 that returns to scale are given by

$$\epsilon(1, x) = \sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i} \frac{x_i}{f(x)}$$

$$\Rightarrow \sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i} x_i = \epsilon(1, x) f(x) \tag{93}$$

For a homogeneous function (equation 78) the Euler equation implies

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} x_i = r f(x) \tag{94}$$

where r is the degree of homogeneity. Combining the results in equations 93 and 94 we obtain

$$\epsilon(1, x) = r \tag{95}$$

Alternatively it can be shown directly by using the definition of homogeneity and the definition of ϵ. First note that if a function is homogeneous $f(\lambda x) = \lambda^r f(x)$. Now write the definition of the elasticity of scale, substitute $\lambda^r f(x)$ for $f(\lambda x)$ and then take the derivative.

$$
\begin{aligned}
\epsilon(\lambda, x) &= \frac{\partial \ln f(\lambda x)}{\partial \ln \lambda} = \frac{\partial f(\lambda x)}{\partial \lambda} \frac{\lambda}{f(\lambda x)} \\
&= \frac{\partial[\lambda^{r} {}^{f}(x)]}{\partial \lambda} \frac{\lambda}{\lambda^{r} {}^{f}(x)} \\
&= r \lambda^{r-1} f(x) \cdot \frac{\lambda}{\lambda^{r} f(x)} \\
&= r
\end{aligned}
\tag{96}
$$

This means that the isoquants spread or bunch at a constant rate and are parallel. Figure 37 shows the isoquants for a CES production function that is homogeneous of degree one.

FIGURE 37. Level Sets for CES Production Function that is Homogeneous of Degree 1
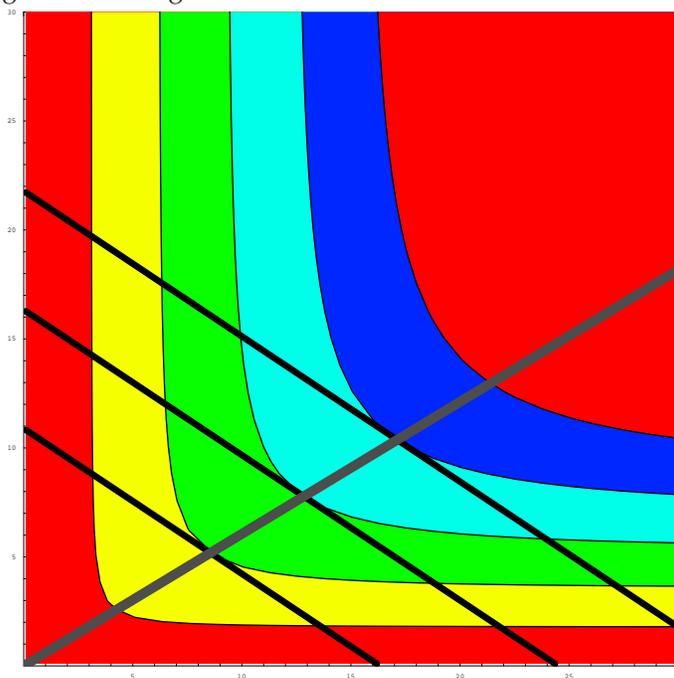
Figures 38 and 39 show the isoquants for production functions that are homogeneous of a degree less than one.

Figure 40 shows the isoquant for a production function that is homogeneous of a degree greater than one.

FIGURE 38. Level Sets for Production Function that is Homogeneous of Degree Less than 1
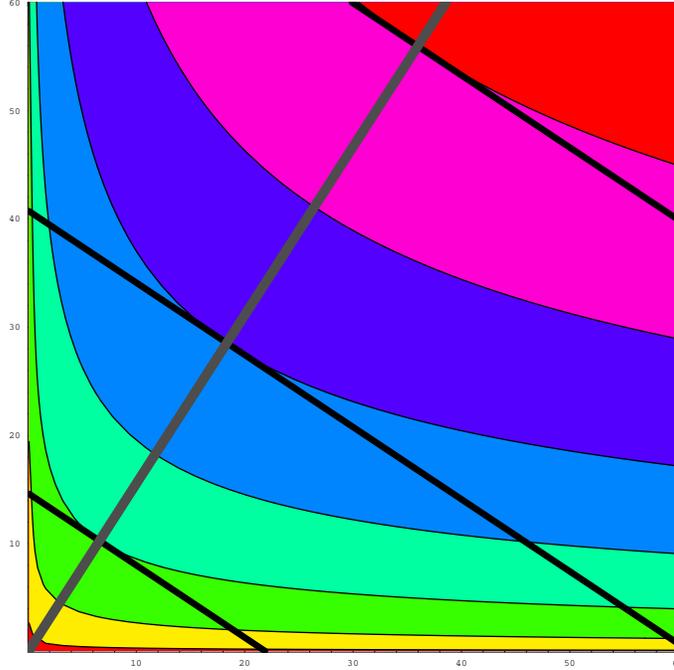
7. HOMOTHETIC FUNCTIONS

7.1. **Definition of a Homothetic Function.** A function f(x, z) is homothetic in x if it can be written

$$y = f(x, z) = H(x, z) = F(\phi(x, z), z) \tag{97}$$

where $z \in \mathrm{R}^m_+$ and F is continuously differentiable to the second degree (C^2), finite, non-negative and non-decreasing. Furthermore

$(a) \ \lim_{\mu \to \infty} F(\mu, \, z) \ = \ \infty$

$(b) \ F(0, z) \ = \ 0$

$(c) \ \phi(x, z)$ is positively linear homogeneous (PLH) in x

7.2. **Alternative Definition of a Homothetic Function.** Let y = f(x)= F(ϕ(x)) where F has the same properties as in the original definition. Then consider the function h(y) = ϕ(x) where ϕ(x) is PLH and h( ) = F^{-1}( ) and it is assumed that F^{-1} exists. The idea is that a transformation of y is homogeneous. Specifically the function in equation 98 is a homothetic function.

$$h(y) = \phi(x), \ \text{where} \ \phi(x) \ \text{is positively linear homogeneous (PLH) in x} \tag{98}$$

7.3. **Some Properties of Homothetic Functions.**

FIGURE 39. Level Sets for CES Production Function that is Homogeneous of Degree Less than 1
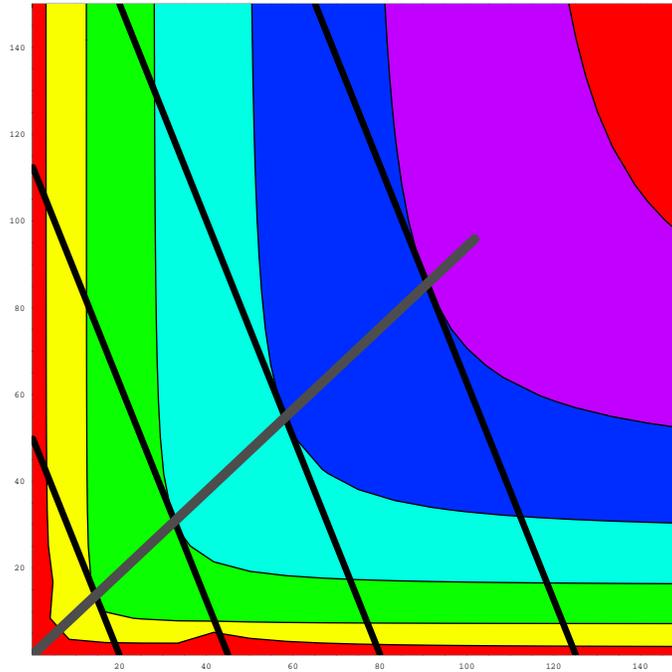
7.3.1. *Rate of technical substitution for homothetic functions.* Define y as follows

$$y = h(x) = F(\phi(x)) \tag{99}$$

Using the implicit function theorem (equation 43), find $\frac{\partial x_i}{\partial x_j}$.

$$
\begin{aligned}
\frac{\partial x_i}{\partial x_j} &= \frac{-\frac{\partial F}{\partial x_j}}{\frac{\partial F}{\partial x_i}} \\
&= -\frac{F'\frac{\partial \phi}{\partial x_j}}{F'\frac{\partial \phi}{\partial x_i}} \\
&= -\frac{\frac{\partial \phi}{\partial x_j}}{\frac{\partial \phi}{\partial x_i}}
\end{aligned}
\tag{100}
$$

ϕ is homogeneous of degree 1 so its derivatives are homogeneous of degree zero by the results in section 6.2. The ratio of these derivatives is also homogeneous of degree zero so that the rate of technical substitution does not change as x changes; i.e. the isoquants are parallel because RTS is constant along a ray from the origin just as with positively linear homogeneous functions. Lau [14] also shows that this condition is sufficient for a production function to be homothetic. This implies that we can obtain the isoquant for any y ≥ 0 by a radical expansion of the unit

FIGURE 40. Level Sets for Production Function that is Homogeneous of Degree Greater than 1
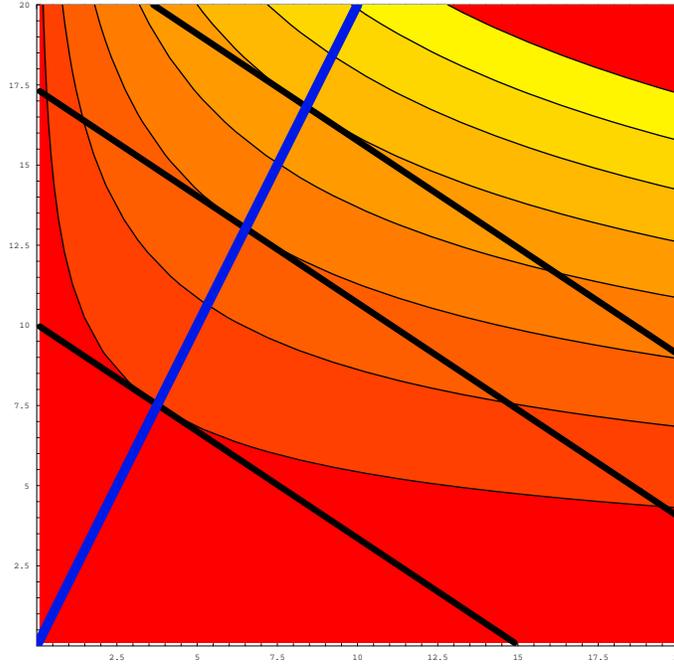
isoquant in the fixed ratio h(y)/h(1). Figure 41 shows the isoquants for a Cobb-Douglas production function that is homogeneous of degree one.

$$y \; = \; \left(5 \, x_1^{.25} \, x_2^{.75}\right) \tag{101}$$

Note that the level sets are evenly spaced $\{20, 40, 60, 80, 100, 120\}$. Also notice that along a ray through the origin the tangent lines are parallel. We can construct the level set for all levels of output by expanding the level set for any one level of output inward or outward.

Now let's consider a transformation of a linear homogeneous CES production function to create a homothetic function.

$$
\begin{aligned}
f(x) \; &= \; \left[0.3 \, x_1^{-2} \; + \; 0.1 \, x_2^{-2}\right]^{\frac{-1}{2}} \\
y = F[f(x)] \; &= \; f(x)^2 \; + \; f(x) \\
&= \; \left[\left[0.3 \, x_1^{-2} \; + \; 0.1 \, x_2^{-2}\right]^{\frac{-1}{2}}\right]^2 \; + \; \left[0.3 \, x_1^{-2} \; + \; 0.1 \, x_2^{-2}\right]^{\frac{-1}{2}}
\end{aligned}
\tag{102}
$$

The function is shown in figure 42 while the level sets are contained in figure 43. The function is not concave and has increasing returns to scale. The level sets are convex and parallel but become closer together for equal increases in output.

FIGURE 41. Level Sets for Cobb-Douglas Prodcution Function that is Homogeneous of Degree 1
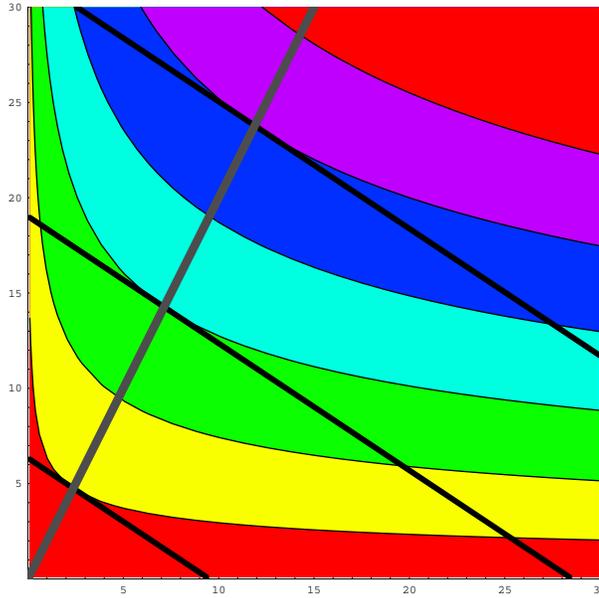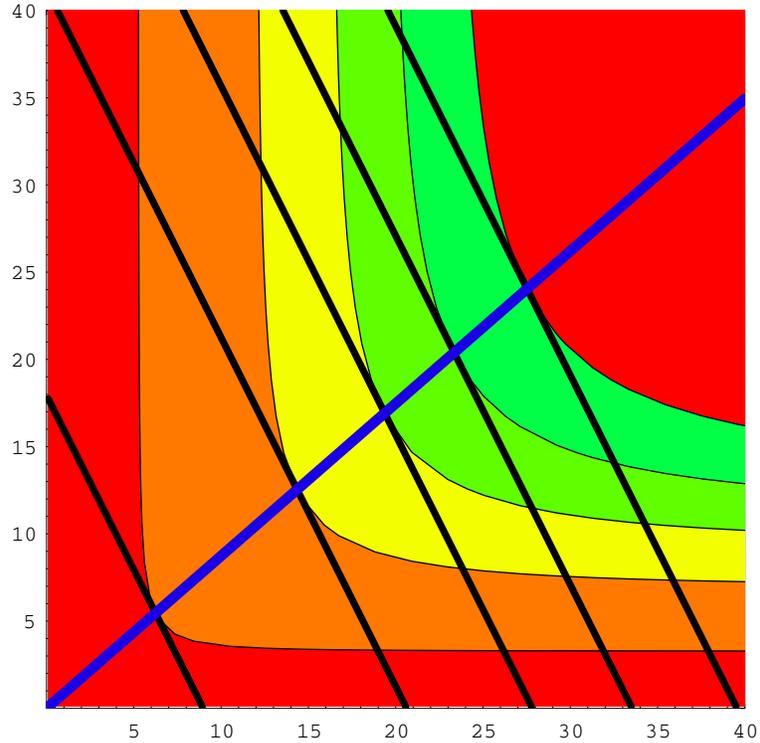

The spacing depends on the level of output and is not a fixed rate of increase or decrease.

FIGURE 42. Homothetic Production Function
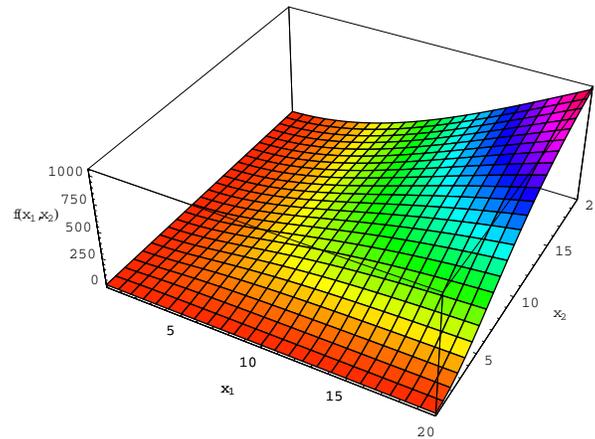
7.3.2. *Elasticity of scale for homothetic functions.* Let $y = f(x) = F(\phi(x))$ or $h(y) = \phi(x)$ as in equation 98.

FIGURE 43. Level Sets for Homothetic Function with Increasing Returns to Scale

$h(y) = \phi(x)$, where $\phi(x)$ is positively linear homogeneous (PLH) in x

Differentiate both sides of the identity with respect to x_i

$$h'(y)\,\frac{\partial y}{\partial x_i} \;=\; \frac{\partial \phi}{\partial x_i} \;\Rightarrow\; \frac{\partial y}{\partial x_i} \;=\; \frac{\frac{\partial \phi}{\partial x_i}}{h'(y)} \tag{103}$$

Then write the definition of ϵ and substitute

$$\epsilon(1,\,x) \;=\; \sum_{i=1}^{n} \frac{\partial F(\phi(x))}{\partial x_i} \,\cdot\, \frac{x_i}{F(\phi(x))}$$

$$=\; \sum_{i=1}^{n} \frac{\partial y}{\partial x_i} \,\cdot\, \frac{x_i}{y}$$

$$=\; \sum_{i=1}^{n} \frac{\frac{\partial \phi}{\partial x_i}}{h\prime(y)} \frac{x_i}{y} \tag{104}$$

$$=\; \frac{h\,(y)}{h\prime(y)\,y} \; by \; Euler's \; Theorem$$

$$=\; \tilde{\epsilon}(y)$$

so ϵ is a function only of y and not a function of x. This is clear from figure 43.

7.4. A Generalized Euler Equation for Homothetic Functions.

Let $y = f(x) = F(\phi(x))$ or alternatively let $h(y) = \phi(x)$. Then from equation 104 we know that

$$\epsilon \;=\; \frac{h(y)}{h'(y)y} \tag{105}$$

where

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{x_i}{y} \;=\; \frac{h(y)}{h\prime(y)\,y} \;=\; \epsilon(y) \tag{106}$$

This implies then that

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, x_i \;=\; \epsilon(y)\, y \tag{107}$$

So ϵ depends on y and not on the mix of the x's. With homogeneous functions ϵ is a constant not depending on y.

7.5. Multiplication property of homothetic functions.

If h represents a homothetic function then

Theorem 1.

$$h(y) \;=\; x_i \,\cdot\, V\!\left(\frac{\tilde{x}}{x_i}\right) \;=\; x_i\, V(\psi) \tag{108}$$

where $\tilde{x}$ is $(x_1, x_2, x_{i-1}, x_{i+1}, \ldots, x_n)$ and

$$\psi_j \;=\; \frac{x_j}{x_i}, \quad j \neq i \tag{109}$$

Here h is a transform like F, in fact $h(z) = F^{-1}(z)$ and V is a nondecreasing concave function.

Proof. Let $y = F(\phi(x))$ where $\phi(x)$ is PLH. Then

$$
\begin{aligned}
y \ &= F(\phi(x)) \\
&= F(\, x_i \cdot \phi[\psi_1, \ \psi_2 \ \dots \ 1 \ \dots \ \psi_n]) \ \ \text{(by PLH)} \\
\Rightarrow \ F^{-1}(y) \ &= x_i \cdot \phi[\psi_1, \ \psi_2 \ \dots \ 1 \ \dots \ \psi_n]) \\
\Rightarrow \ x_i \ &= \frac{F^{-1}(y)}{\phi(\psi_1, \ \psi_2 \dots 1 \dots \psi_n)} \\
&= \frac{h\,(y)}{V\,(\psi)} \\
\Rightarrow h(y) \ &= \ x_i \, V(\psi)
\end{aligned}
\tag{110}
$$

$\square$

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