SIMPLE CONSTRAINED OPTIMIZATION

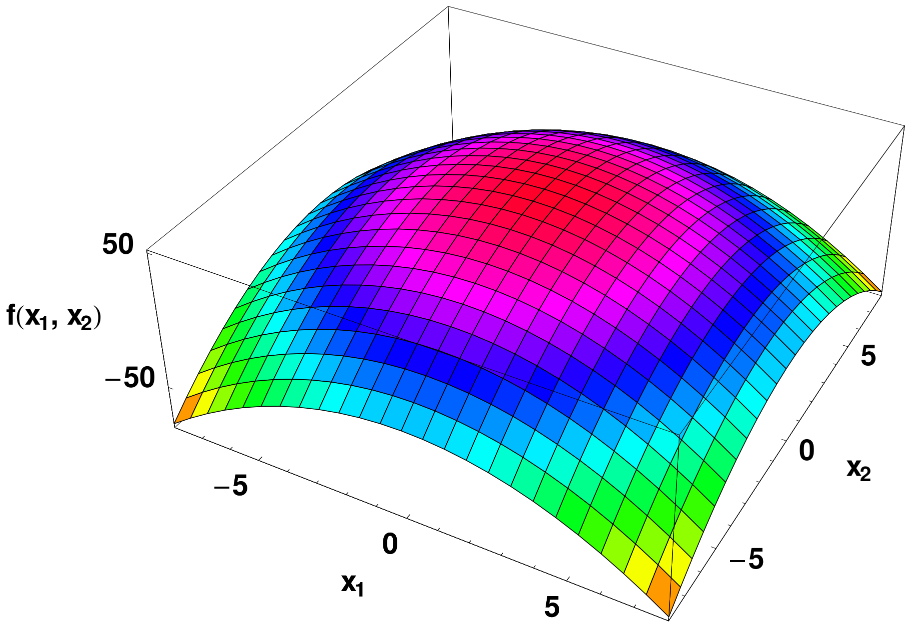
1. Intuitive Introduction to Constrained Optimization

Consider the following function which has a maximum at the origin.

$$y = f(x_1, x_2) = 49 - x_1^2 - x_2^2 \tag{1}$$

The graph is contained in figure 1.

FIGURE 1. The function $y = 49 - x_1^2 - x_2^2$



The tangent plane to the graph at the origin is shown in figure 2. Now consider only values of x_1 and x_2 that satisfy the equation

$$x_1 + 3x_2 - 10 = 0 \tag{2}$$

Above this line in the x_1-x_2 plane is an infinity of points. We can construct a plane above this line in R^3. This plane is shown in figure 3.

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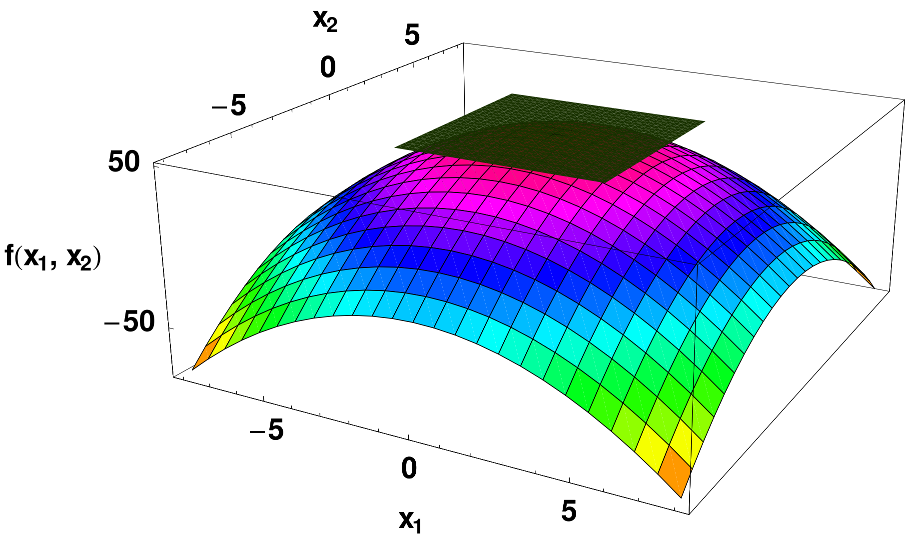
FIGURE 2. Tangent plane to the function $y = 49 - x_1^2 - x_2^2$



Now consider maximizing the function $y = 49 - x_1^2 - x_2^2$ subject to the condition that the values of x_1 and x_2 chosen lie on the line $x_1 + 3x_2 - 10 = 0$. Graphically we want to pick points on the surface that also lie on the plane above the line. Both the function and the plane on which we must pick the points are shown in figure 4.

From inspection of the graph, it is clear that the maximum of the function that also lies in the plane is less than the global maximum of the function. If we look at a closeup of the graph (Figure 5) in the vicinity of what seems to be the constrained maximum, we can visually guess at corresponding values of x_1 and x_2.

The constrained maximum looks to be somewhere around $x_1 = 1$ and $x_2 = 3$. This of course is quite a distance from the unconstrained maximum of (0, 0). As will be shown in section 2, the maximal value of the function is at $x_1 = 1$ and $x_2 = 3$ as can be seen in figure 6.

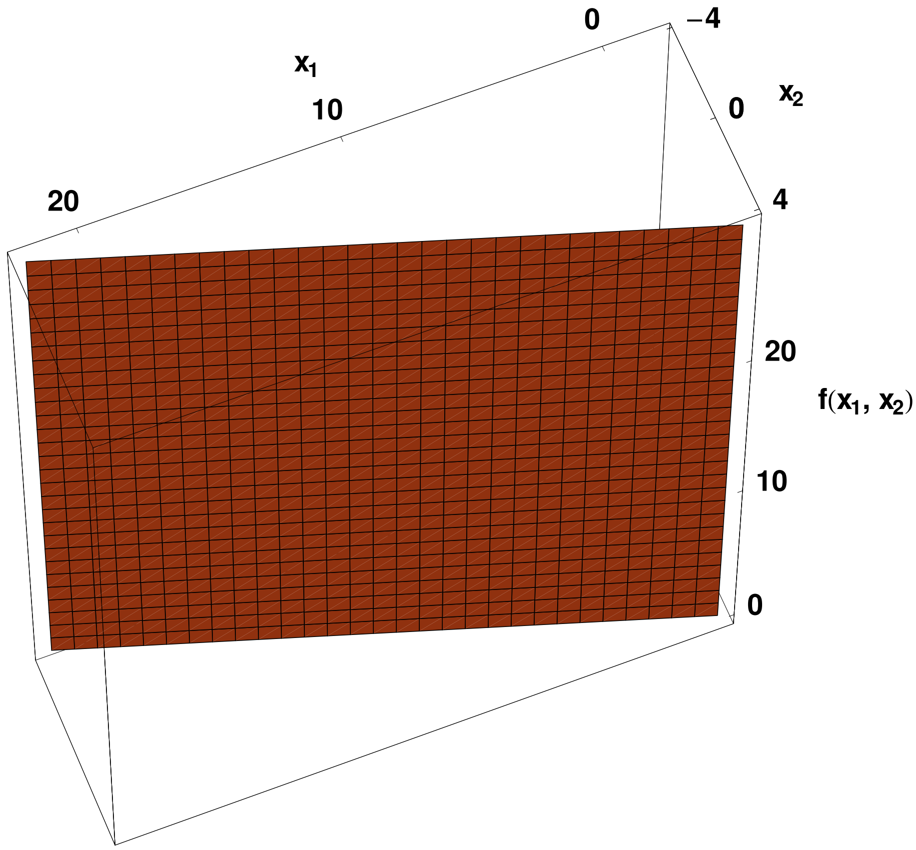
We can also graph the level curves of the function and the constraint as in figure 7. Level curves are indicated at values of 14, 21, 28, 35, 42 and 49. The constraint is the straight darker line in the figure.

It seems clear that the constrained maximum must be between the level curves for 35 and 42. Adding a level curve for y = 39, we can see the optimum in figure 8.

Thus we have found a constrained maxima for the function using graphical methods. A few things seem to characterize the extreme point.

1: The extreme point lies on the surface but above a point in x_1-x_2 space that satisfies the constraint.
2: At the constrained extreme point, the constraint and the level surfaces of the function are tangent.

FIGURE 3. Points in R^3 satisfying the equation $x_1 + 3x_2 - 10 = 0$



2. FORMAL ANALYSIS OF CONSTRAINED OPTIMIZATION PROBLEMS

2.1. **Formal setup of the constrained optimization problem.** Consider the problem defined by

$$\max_{x_1, x_2} f(x_1, x_2) \text{ subject to } g(x_1, x_2) = 0. \tag{3}$$

where $g(x_1, x_2) = 0$ denotes a constraint on the values of x_1 and x_2.

2.2. **Solution by substitution.** One method that sometimes works is to solve the constraint equation for x_1 in terms of x_2 and then substitute in $f(x_1, x_2)$. For the problem at hand this would yield

$$\begin{aligned} x_1 + 3x_2 - 10 &= 0 \\ \Rightarrow x_1 &= 10 - 3x_2 \end{aligned} \tag{4}$$

If we rewrite the function f using this substitution we obtain

FIGURE 4. The function $y = 49 - x_1^2 - x_2^2$ and the constraint $x_1 + 3x_2 - 10 = 0$

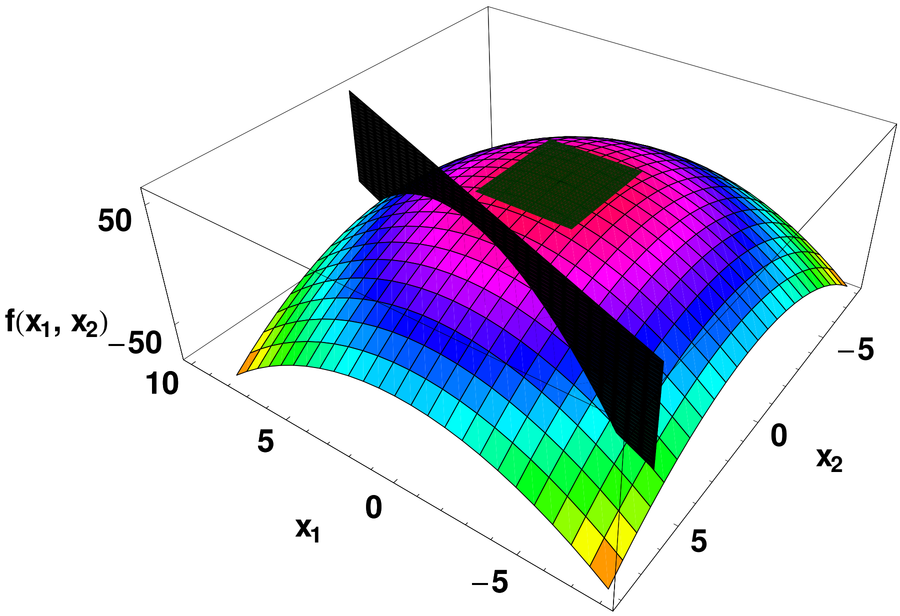
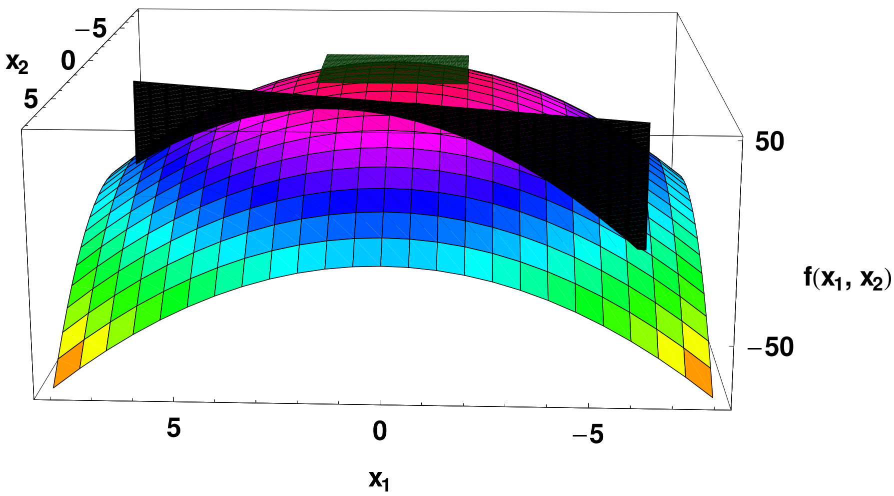
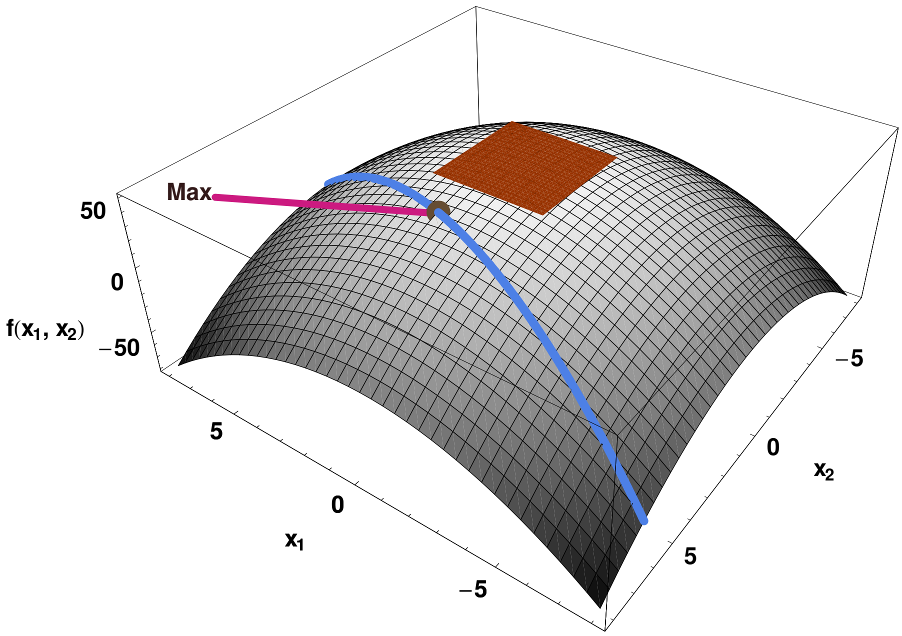


FIGURE 5. Alternative view of $y = 49 - x_1^2 - x_2^2$ and $x_1 + 3x_2 - 10 = 0$



$$
\begin{aligned}
y \;=\; f(x_1, x_2) &= 49 \;-\; x_1^2 \;-\; x_2^2 \\
&= 49 \;-\; (10 \;-\; 3x_2)^2 \;-\; x_2^2 \\
&= 49 \;-\; (100 \;-\; 60x_2 + 9x_2^2) \;-\; x_2^2 \\
&= -\,51 \;+\; 60x_2 - 10x_2^2
\end{aligned} \tag{5}
$$

FIGURE 6. Maximum of $y = 49 - x_1^2 - x_2^2$ subject to $x_1 + 3x_2 - 10 = 0$



Now we can maximize this function by taking the derivative with respect to x_2, setting this equation equal to zero, and then solving for x_2 as follows

$$
\begin{aligned}
y &= -51 + 60x_2 - 10x_2^2 \\
\frac{dy}{dx_2} &= 60 - 20x_2 = 0 \\
\Rightarrow 20x_2 &= 60 \\
\Rightarrow x_2 &= 3
\end{aligned} \tag{6}
$$

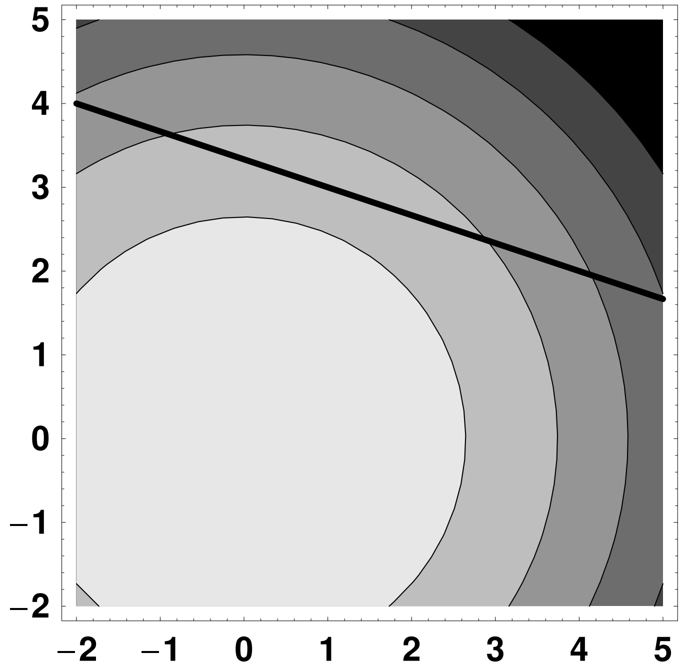
Substituting in equation 4 we obtain

$$
\begin{aligned}
x_1 &= 10 - 3x_2 \\
&= 10 - (3)(3) \\
&= 10 - 9 = 1
\end{aligned} \tag{7}
$$

We can check to see that this is maximum by looking at the second derivative of y in equation 6. This will give

$$
\begin{aligned}
\frac{dy}{dx_2} &= 60 - 20x_2 \\
\frac{d^2y}{dx_2^2} &= -20
\end{aligned} \tag{8}
$$

FIGURE 7. Level curves of the function $y = 49 - x_1^2 - x_2^2$ and constraint $x_1 + 3x_2 - 10 = 0$



At the $x_2 = 3$ this is negative, and so we have a maximum.

While the method of substitution will work in many cases it breaks down in a number of situations. For example if there is no explicit way to solve the constraint for one of the variables, there is no easy way to make the substitution in f. In some situations, if we choose the wrong variable to solve out of the constraint we may end up with a point that maximizes or minimizes f but does not satisfy the constraint. And extending the method of substitution to multiple variables is often difficult. We thus turn to another method due to Lagrange and make use of the fact that the level curves of the function and the constraint are tangent to one another.

2.3. **The method of Lagrange.**

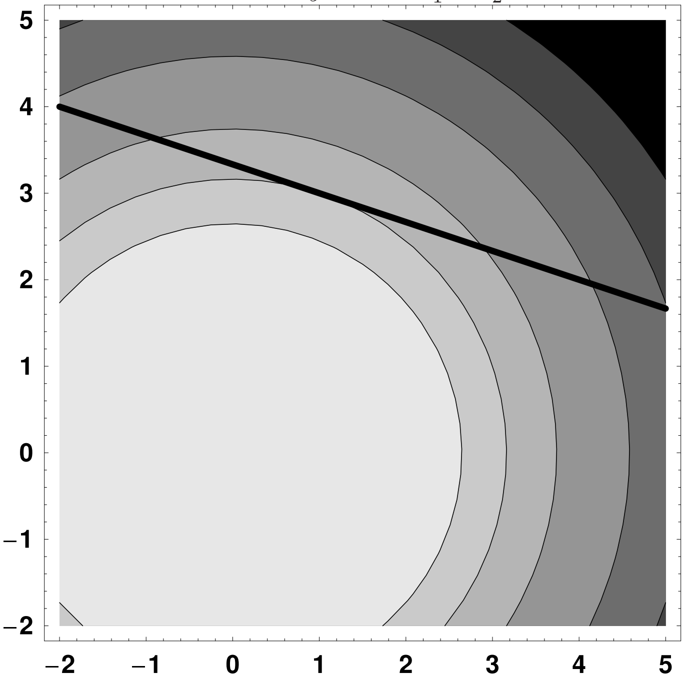
2.3.1. *The Lagrangian.* The solution to a constrained optimization problem is obtained by finding the critical values of the Lagrangian function

$$\mathcal{L}(x_1,\, x_2,\, \lambda) \;=\; f(x_1,\, x_2) \;-\; \lambda\, g(x_1,\, x_2) \tag{9}$$

Notice that the gradient of $\mathcal{L}$ with respect to x_1 and x_2 will involve a set of derivatives that looks like this

$$\nabla \mathcal{L}(x_1,\, x_2;\; \lambda) \;=\; \begin{pmatrix} \frac{\partial f}{\partial x_1} \;-\; \lambda\, \frac{\partial g}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \;-\; \lambda\, \frac{\partial g}{\partial x_2} \end{pmatrix} \tag{10}$$

FIGURE 8. More level curves of the function $y = 49 - x_1^2 - x_2^2$ and constraint $x_1 + 3x_2 - 10 = 0$



2.3.2. *Necessary first order conditions for an extreme point.* The necessary conditions for an extremum of f(x_1, x_2)with the equality constraints g(x_1, x_2) = 0 are

$$\nabla \mathcal{L}(x_1^*, x_2^*, \lambda^*) = 0 \tag{11}$$

This, of course implies that

$$\nabla \mathcal{L}(x_1, x_2, \lambda) = \begin{pmatrix} \frac{\partial f}{\partial x_1} - \lambda \frac{\partial g}{\partial x_1} \\ \frac{\partial f}{\partial x_2} - \lambda \frac{\partial g}{\partial x_2} \\ -g(x_1, x_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{12}$$

2.3.3. *Simple example.* For the example problem the Lagrangian is as follows

$$\begin{aligned} \mathcal{L}(x_1, x_2, \lambda) &= f(x_1, x_2) - \lambda g(x_1, x_2) \\ &= 49 - x_1^2 - x_2^2 - \lambda (x_1 + 3x_2 - 10) \end{aligned} \tag{13}$$

Taking the partial derivatives with respect to x_1, x_2, and λ we obtain

$$\mathcal{L}(x_1, x_2, \lambda) = 49 - x_1^2 - x_2^2 - \lambda(x_1 + 3x_2 - 10)$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = -2x_1 - \lambda = 0 \tag{14a}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = -2x_2 - 3\lambda = 0 \tag{14b}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -x_1 - 3x_2 + 10 = 0 \tag{14c}$$

Equation 14a can be solved for λ yielding

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= -2x_1 - \lambda = 0 \\ \Rightarrow 2x_1 &= -\lambda \\ \Rightarrow x_1 &= \frac{-\lambda}{2} \end{aligned} \tag{15}$$

Similarly equation 14b can be solved λ implying

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_2} &= -2x_2 - 3\lambda = 0 \\ \Rightarrow 2x_2 &= -3\lambda \\ \Rightarrow x_2 &= \frac{-3\lambda}{2} \end{aligned} \tag{16}$$

Substituting equations 15 and 16 into equation 14c will allow us to solve for λ as follows

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \lambda} &= -x_1 - 3x_2 + 10 = 0 \\ \Rightarrow -\left(\frac{-\lambda}{2}\right) - (3)\left(\frac{-3\lambda}{2}\right) &= -10 \\ \Rightarrow \left(\frac{\lambda}{2}\right) + \left(\frac{9\lambda}{2}\right) &= -10 \\ \Rightarrow \frac{10}{2}\lambda &= -10 \\ \Rightarrow 5\lambda &= -10 \\ \Rightarrow \lambda &= -2 \end{aligned} \tag{17}$$

Now substituting in equations 15 and 16 for x_1 and x_2 we obtain

$$\begin{aligned} x_1 &= \frac{-\lambda}{2} = \frac{-(-2)}{2} = 1 \\ x_2 &= \frac{-3\lambda}{2} = \frac{(-3)(-2)}{2} = \frac{6}{2} = 3 \end{aligned} \tag{18}$$

Thus we obtain the same answer as with substitution.

2.4. **Sufficient conditions for a constrained extremum problem.** The sufficient conditions for the two variable case with one constraint will be stated without proof at this time.

2.4.1. *Sufficient conditions for a maximum.*

$$\det \begin{bmatrix} \frac{\partial^2 \mathfrak{L}(x^*,\,\lambda^*)}{\partial x_1 \partial x_1} & \frac{\partial^2 \mathfrak{L}(x^*,\,\lambda^*)}{\partial x_1 \partial x_2} & \frac{\partial g(x^*)}{\partial x_1} \\ \frac{\partial^2 \mathfrak{L}(x^*,\,\lambda^*)}{\partial x_2 \partial x_1} & \frac{\partial^2 \mathfrak{L}(x^*,\,\lambda^*)}{\partial x_2 \partial x_2} & \frac{\partial g(x^*)}{\partial x_2} \\ \frac{\partial g(x^*)}{\partial x_1} & \frac{\partial g(x^*)}{\partial x_2} & 0 \end{bmatrix} > 0 \tag{19}$$

2.4.2. *Sufficient conditions for a minimum.*

$$\det \begin{bmatrix} \frac{\partial^2 L(x^*,\,\lambda^*)}{\partial x_1 \partial x_1} & \frac{\partial^2 \mathfrak{L}(x^*,\,\lambda^*)}{\partial x_1 \partial x_2} & \frac{\partial g(x^*)}{\partial x_1} \\ \frac{\partial^2 \mathfrak{L}(x^*,\,\lambda^*)}{\partial x_2 \partial x_1} & \frac{\partial^2 \mathfrak{L}(x^*,\,\lambda^*)}{\partial x_2 \partial x_2} & \frac{\partial g(x^*)}{\partial x_2} \\ \frac{\partial g(x^*)}{\partial x_1} & \frac{\partial g(x^*)}{\partial x_2} & 0 \end{bmatrix} < 0 \tag{20}$$

3. EXAMPLES OF CONSTRAINED OPTIMIZATION

3.1. **Minimizing the cost of obtaining a fixed level of output.**

3.1.1. *Production function.* Consider a production function given by

$$y = 20x_1 - x_1^2 + 15x_2 - x_2^2 \tag{21}$$

3.1.2. *Prices and constraint.* Let the prices of x_1 and x_2 be 10 and 5 respectively. Then minimize the cost of producing 55 units of output given these prices. The objective function is $10x_1 + 5x_2$. The constraint is $20x_1 - x_1^2 + 15x_2 - x_2^2 = 55$.

3.1.3. *Setting up the Lagrangian function and solving for the optimal values.* The Lagrangian is given by

$$\mathcal{L} = 10x_1 + 5x_2 - \lambda(20x_1 - x_1^2 + 15x_2 - x_2^2 - 55)$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = 10 - \lambda(20 - 2x_1) = 0 \tag{22a}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 5 - \lambda(15 - 2x_2) = 0 \tag{22b}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = (-1)(20x_1 - x_1^2 + 15x_2 - x_2^2 - 55) = 0 \tag{22c}$$

If we take the ratio of the first two first order conditions in equation 22 we obtain

$$\begin{aligned} \frac{10}{5} &= 2 = \frac{20 - 2x_1}{15 - 2x_2} \\ \Rightarrow 30 - 4x_2 &= 20 - 2x_1 \\ \Rightarrow 10 - 4x_2 &= -2x_1 \\ \Rightarrow x_1 &= 2x_2 - 5 \end{aligned} \tag{23}$$

Now plug this result into the negative of equation 22c to obtain

$$20(2x_2 - 5) - (2x_2 - 5)^2 + 15x_2 - x_2^2 - 55 = 0 \tag{24}$$

Multiplying out and solving for x_2 will give

$$\begin{aligned} 40x_2 - 100 - (4x_2^2 - 20x_2 + 25) + 15x_2 - x_2^2 - 55 &= 0 \\ \Rightarrow 40x_2 - 100 - 4x_2^2 + 20x_2 - 25 + 15x_2 - x_2^2 - 55 &= 0 \\ \Rightarrow -5x_2^2 + 75x_2 - 180 &= 0 \\ \Rightarrow 5x_2^2 - 75x_2 + 180 &= 0 \\ \Rightarrow x_2^2 - 15x_2 + 36 &= 0 \end{aligned} \tag{25}$$

Now solve this quadratic equation for x_2 as follows

$$\begin{aligned} x_2 &= \frac{15 \pm \sqrt{225 - 4(36)}}{2} \\ &= \frac{15 \pm \sqrt{81}}{2} \\ &= 3 \text{ or } 12 \end{aligned} \tag{26}$$

Given that we know the objective is to minimize cost we would likely choose 3. Therefore,

$$\begin{aligned} x_1 &= 2x_2 - 5 \\ &= 1 \end{aligned} \tag{27}$$

For x_2=12, x_1 = 19. The minimum cost is obtained by substituting into the cost expression to obtain

$$C = 10(1) + 5(3) = 25 \tag{28}$$

The Lagrangian multiplier λ can be obtained by substituting for x_1 in equation 22a.

$$\begin{aligned} 10 - \lambda(20 - 2(1)) &= 0 \\ \rightarrow \lambda = \frac{10}{18} = \frac{5}{9} &= 0.5\overline{5} \end{aligned} \tag{29}$$

3.1.4. *Sufficient conditions.* In order to check which pair of x values is a minimum and which is a maximum, we need to form the bordered Hessian matrix. Remember that the Lagrangian for this problem is given by

$$\mathfrak{L} = 10x_1 + 5x_2 - \lambda(20x_1 - x_1^2 + 15x_2 - x_2^2 - 55) \tag{30}$$

The relevant derivatives are

$$\frac{\partial \mathfrak{L}(x_1, x_2, \lambda)}{\partial x_1} = 10 - \lambda(20 - 2x_1) \tag{31a}$$

$$\frac{\partial \mathfrak{L}(x_1, x_2, \lambda)}{\partial x_2} = 5 - \lambda(15 - 2x_2) \tag{31b}$$

$$\frac{\partial \mathfrak{L}(x_1, x_2, \lambda)}{\partial \lambda} = (-1)(20x_1 - x_1^2 + 15x_2 - x_2^2 - 55) \tag{31c}$$

$$\frac{\partial^2 \mathfrak{L}(x_1, x_2, \lambda)}{\partial x_1 \partial x_1} = 2\lambda, \qquad \frac{\partial^2 \mathfrak{L}(x_1, x_2, \lambda)}{\partial x_2 \partial x_1} = 0 \tag{31d}$$

$$\frac{\partial^2 \mathfrak{L}(x_1, x_2, \lambda)}{\partial x_1 \partial x_2} = 0, \qquad \frac{\partial^2 \mathfrak{L}(x_1, x_2, \lambda)}{\partial x_2 \partial x_2} = 2\lambda \tag{31e}$$

$$\frac{\partial g(x_1, x_2)}{\partial x_1} = 20 - 2x_1, \qquad \frac{\partial g(x_1, x_2)}{\partial x_2} = 15 - 2x_2 \tag{31f}$$

This gives for the bordered Hessian matrix

$$\begin{bmatrix} 2\lambda & 0 & 20 - 2x_1 \\ 0 & 2\lambda & 15 - 2x_2 \\ 20 - 2x_1 & 15 - 2x_2 & 0 \end{bmatrix} \tag{32}$$

At the root x = (1,3), this gives

$$\begin{bmatrix} \frac{10}{9} & 0 & 18 \\ 0 & \frac{10}{9} & 9 \\ 18 & 9 & 0 \end{bmatrix} \tag{33}$$

The determinant is

$$\begin{aligned}\left(\frac{10}{9}\right)\left(\frac{10}{9}\right)(0) \,+\, (0)(9)(18) \,+\, (18)(0)(9) - \left((18)\left(\frac{10}{9}\right)(18) \,+\, (0)(0)(0) \,+\, \left(\frac{10}{9}\right)(9)(9)\right)& \\ = 0 \,+\, 0 \,+\, 0 \,-\, (360 \,+\, 0 \,+\, 90)& \\ = \,-\,450&\end{aligned}$$

Given that this is negative, the values x_1 = 1 and x_2 = 3 then minimize cost.

Now consider the other two roots, x_1 = 19 and x_2 = 12, λ = -$\frac{5}{9}$. The bordered Hessian is

$$\begin{bmatrix} 2\lambda & 0 & 20 \,-\, 2x_1 \\ 0 & 2\lambda & 15 \,-\, 2x_2 \\ 20 \,-\, 2x_1 & 15 \,-\, 2x_2 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{10}{9} & 0 & -18 \\ 0 & -\frac{10}{9} & -9 \\ -18 & -9 & 0 \end{bmatrix} \tag{34}$$

This has determinant

$$\begin{aligned}\left(-\frac{10}{9}\right)\left(-\frac{10}{9}\right)(0) \,+\, (0)(-9)(-18) \,+\, (-18)(0)(-9) - \left((-18)\left(-\frac{10}{9}\right)(-18) \,+\, (0)(0)(0) \,+\, \left(-\frac{10}{9}\right)(-9)(-9)\right)& \\ = 0 \,+\, 0 \,+\, 0 \,-\, (-360 \,+\, 0 \,+\, -90)& \\ = 450&\end{aligned}$$

Given that this is positive, the values x_1 = 19 and x_2 = 12 then maximize cost.

3.1.5. *Graphical analysis.* The objective function is a plane as shown in figure 9. It is clearly minimized in the positive orthant at the point (0, 0). The level curves are shown in figure 10. If we plot these level curves along with the value of the constraint we obtain a representation as in figure 11. The optimum looks to be somewhere around 1 and 3. We can then refine the diagram as in figure 12.

3.2. **Maximizing a utility function subject to a budget constraint.**

3.2.1. *The utility function.* Consider a utility function given by

$$u \,=\, x_1^{\alpha_1}\, x_2^{\alpha_2} \tag{35}$$

3.2.2. *Prices, income and the budget constraint.* Consider a general problem with prices w_1 and w_2 and income level c_0. The budget constraint is then

$$w_1 x_1 \,+\, w_2 x_2 \,=\, c_0 \tag{36}$$

FIGURE 9. Plane in R^3 representing the equation $10x_1 + 5x_2$

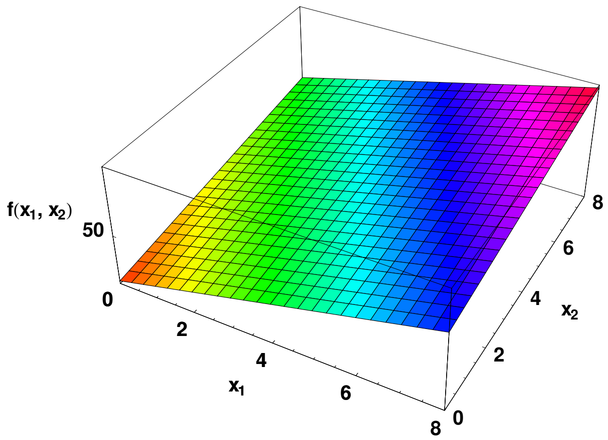
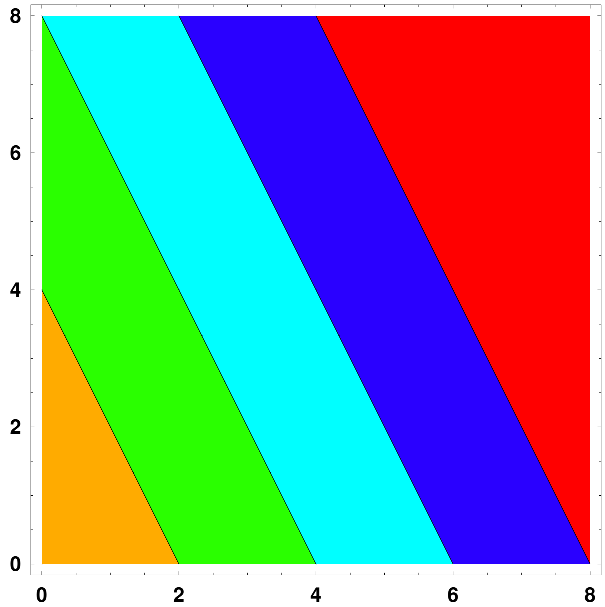


FIGURE 10. Level curves of $10x_1 + 5x_2$

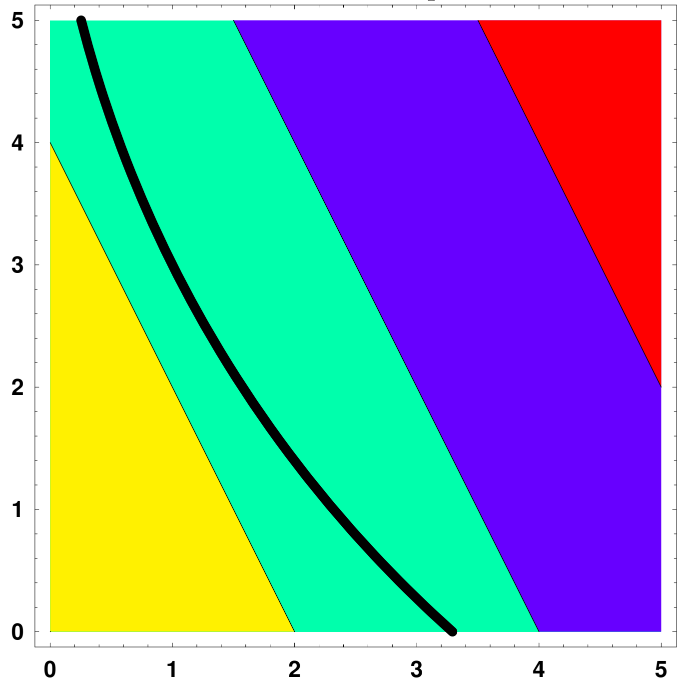


3.2.3. *Setting up the Lagrangian function and solving for the optimal values.* To maximize utility subject to the budget constraint we set up the Lagrangian problem

$$\mathcal{L} = x_1^{\alpha_1} x_2^{\alpha_2} - \lambda [w_1x_1 + w_2x_2 - c_0] \tag{37}$$

The first order conditions are

FIGURE 11. Level curves of $10x_1 + 5x_2$ and production function constraint



$$\frac{\partial \mathcal{L}}{\partial x_1} = \alpha_1\, x_1^{\alpha_1 - 1}\, x_2^{\alpha_2} - \lambda\, w_1 = 0 \tag{38a}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \alpha_2\, x_1^{\alpha_1}\, x_2^{\alpha_2 - 1} - \lambda\, w_2 = 0 \tag{38b}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -\, w_1 x_1 - w_2 x_2 + c_0 = 0 \tag{38c}$$

Taking the ratio of the first and second equations in equation 38 we obtain

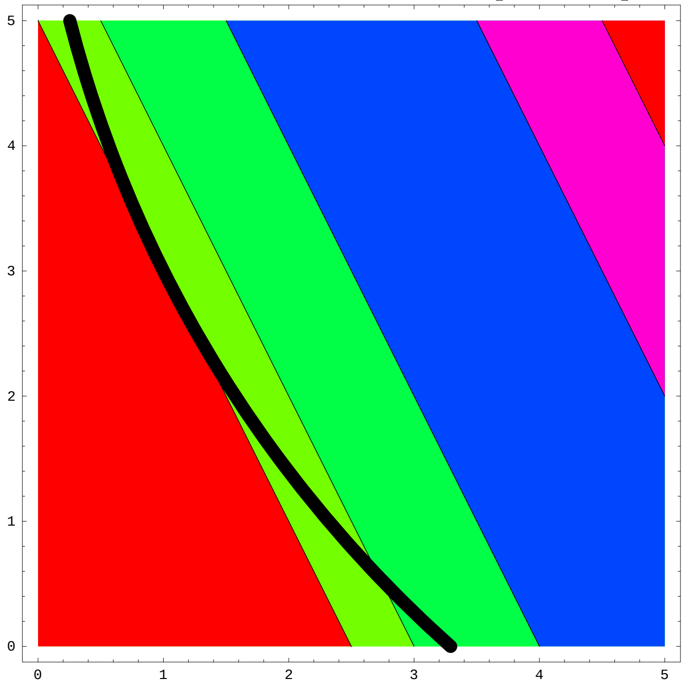
$$\frac{w_1}{w_2} = \frac{\alpha_1 x_2}{\alpha_2 x_1}. \tag{39}$$

We can now solve this equation for x_2 as a function of x_1 and prices. Doing so we obtain

$$x_2 = \frac{\alpha_2 x_1 w_1}{\alpha_1 w_2}. \tag{40}$$

Substituting in the cost equation (38c) we obtain

FIGURE 12. Level curves of $10x_1 + 5x_2$ and optimum input levels



$$
\begin{aligned}
w_1\,x_1 + w_2\,x_2 &= c_0 \\
\Rightarrow\ w_1\,x_1 + w_2\left[\frac{\alpha_2\,x_1\,w_1}{\alpha_1\,w_2}\right] &= c_0 \\
\Rightarrow\ w_1\,x_1 + \left[\frac{\alpha_2\,w_1\,w_2}{\alpha_1\,w_2}\right]\,x_1 &= c_0 \\
\Rightarrow\ w_1\,x_1 + \left[\frac{\alpha_2\,w_1}{\alpha_1}\right]\,x_1 &= c_0 \\
\Rightarrow\ x_1\left[w_1 + \frac{\alpha_2\,w_1}{\alpha_1}\right] &= c_0 \\
\Rightarrow\ x_1\,w_1\left[1 + \frac{\alpha_2}{\alpha_1}\right] &= c_0 \\
\Rightarrow\ x_1\,w_1\left[\frac{\alpha_1 + \alpha_2}{\alpha_1}\right] &= c_0 \\
\Rightarrow\ x_1 = \frac{c_0}{w_1}\left[\frac{\alpha_1}{\alpha_1 + \alpha_2}\right]
\end{aligned}
\tag{41}
$$

We can now obtain x_2 by substitution

$$\begin{aligned} x_2 &= x_1 \left[\frac{\alpha_2 \, w_1}{\alpha_1 \, w_2} \right] \\ &= \frac{c_0}{w_1} \left[\frac{\alpha_1}{\alpha_1 \, + \, \alpha_2} \right] \left[\frac{\alpha_2 \, w_1}{\alpha_1 \, w_2} \right] \\ &= \frac{c_0}{w_2} \left[\frac{\alpha_2}{\alpha_1 \, + \, \alpha_2} \right] \end{aligned} \tag{42}$$

The utility level is obtained by substituting x_1 and x_2 in the utility function

$$\begin{aligned} u &= x_1^{\alpha_1} \, x_2^{\alpha_2} \\ &= \left[\frac{c_0}{w_1} \left(\frac{\alpha_1}{\alpha_1 \, + \, \alpha_2} \right) \right]^{\alpha_1} \left[\frac{c_0}{w_2} \left(\frac{\alpha_2}{\alpha_1 \, + \, \alpha_2} \right) \right]^{\alpha_2} \\ &= \left(\frac{\alpha_1}{\alpha_1 \, + \, \alpha_2} \right)^{\alpha_1} \left(\frac{\alpha_2}{\alpha_1 \, + \, \alpha_2} \right)^{\alpha_2} \left(\frac{c_0}{w_1} \right)^{\alpha_1} \left(\frac{c_0}{w_2} \right)^{\alpha_2} \end{aligned} \tag{43}$$

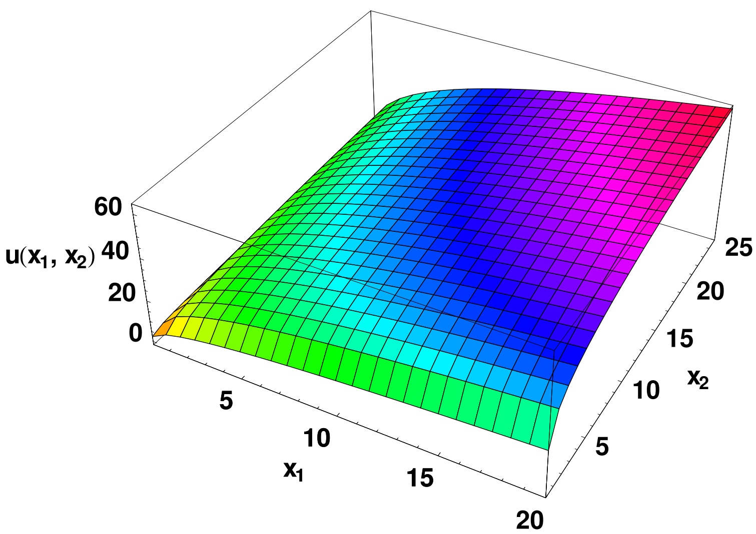
3.3. **Numerical example of maximizing a utility function subject to a budget constraint.**

3.3.1. *The utility function.* Consider a utility function given by

$$u \;=\; 10 x_1^{2/5} \, x_2^{1/5} \tag{44}$$

The utility function is shown in figure 13.

FIGURE 13. Utility Function

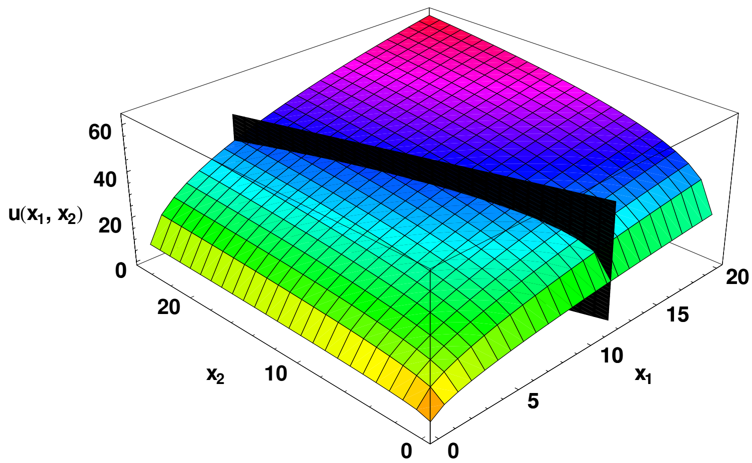


3.3.2. *Prices, income and the budget constraint.* For this problem w_1 = 8 and w_2 =2 and the income level c_0 = 96. The budget constraint is then

$$8\,x_1 \;+\; 2\,x_2 \;=\; 96 \tag{45}$$

The utility function and the constraint are shown in figure 14. A set of level sets (indifference curves) and the budget constraint are shown in figure 15.

FIGURE 14. Utility Function and Constraint



3.3.3. *Setting up the Lagrangian function and solving for the optimal values.* To maximize utility subject to the budget constraint we set up the Lagrangian problem

$$\mathcal{L} \;=\; 10x_1^{2/5}\,x_2^{1/5} \;-\; \lambda\,[\,8\,x_1 \;+\; 2\,x_2 \;-\; 96] \tag{46}$$

The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial x_1} \;=4x_1^{-3/5}\,x_2^{1/5} \;-\; 8\lambda \;=\; 0 \tag{47a}$$

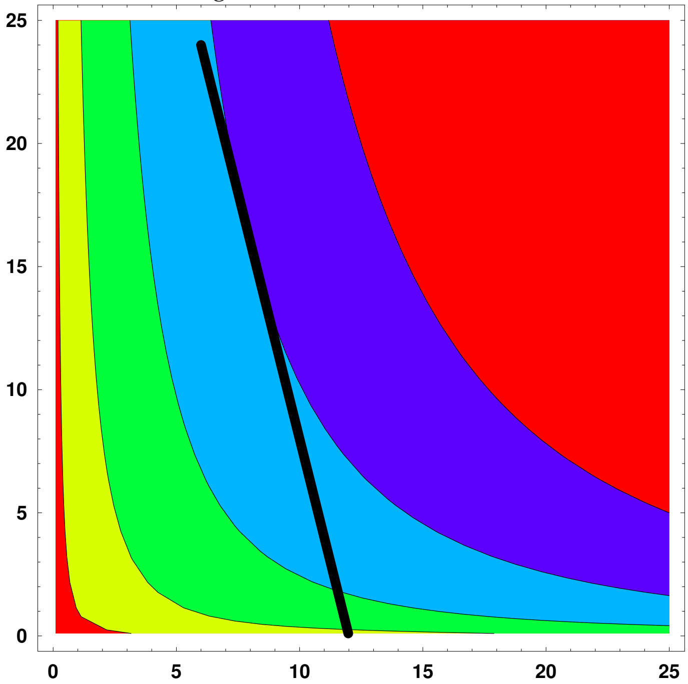
$$\frac{\partial \mathcal{L}}{\partial x_2} \;=2x_1^{2/5}\,x_2^{-4/5} \;-\; 2\lambda \;=\; 0 \tag{47b}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} \;=\; -\,8\,x_1 \;-\; 2\,x_2 \;+\; 96 \;=\; 0 \tag{47c}$$

Taking the ratio of the first and second equations in equation 47 we obtain

$$\begin{aligned} \frac{4x_1^{-3/5}\,x_2^{1/5}}{2x_1^{2/5}\,x_2^{-4/5}} \;&=\frac{8}{2} \;=\; 4 \\ \Rightarrow\; 2x_1^{-1}\,x_2 \;&=4 \end{aligned} \tag{48}$$

FIGURE 15. Budget Constraint and Indifference Curves



We can now solve this equation for x_2 as a function of x_1 and prices. Doing so we obtain

$$x_2 = 2x_1 \tag{49}$$

Substituting in the cost equation (47c) we obtain

$$\begin{aligned} 8x_1 + 2x_2 &= 96 \\ \Rightarrow 8x_1 + 2[2x_1] &= 96 \\ \Rightarrow 8x_1 + 4x_1 &= 96 \\ \Rightarrow 12x_1 &= 96 \\ \Rightarrow x_1 &= 8 \end{aligned} \tag{50}$$

We can now obtain x_2 by substitution

$$\begin{aligned} x_2 &= 2x_1 \\ &= (2)(8) = 16 \end{aligned} \tag{51}$$

The utility level is obtained by substituting x_1 and x_2 in the utility function

$$\begin{aligned} u &= 10x_1^{2/5}\, x_2^{1/5} \\ &= (10)\,(8)^{2/5}\,(16)^{1/5} \\ &= (2)(5)(2^3)^{2/5}\,(2^4)^{1/5} \\ &= (2)(5)(2)^{6/5}\,(2)^{4/5} \\ &= (2)(5)(2)^{10/5} \\ &= (2)(5)(2^2) \\ &= (8)(5) \\ &= 40 \end{aligned} \tag{52}$$

3.3.4. *Sufficient conditions.* In order to check if the x values we obtained lead to a maximum, we need to form the bordered Hessian matrix. Remember that the Lagrangian for this problem is given by

$$\mathcal{L} \;=\; 10x_1^{2/5}\, x_2^{1/5} \;-\; \lambda\,[\,8\,x_1 \;+\; 2\,x_2 \;-\; 96] \tag{53}$$

The relevant derivatives are

$$\frac{\partial \mathcal{L}}{\partial x_1} = 4x_1^{-3/5}\, x_2^{1/5} \;-\; 8\lambda \tag{54a}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 2x_1^{2/5}\, x_2^{-4/5} \;-\; 2\lambda \tag{54b}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} \;=\; -\,8\,x_1 \;-\; 2\,x_2 \;+\; 96 \tag{54c}$$

$$\frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1 \partial x_1} \;=\; -\,\frac{12}{5}x_1^{-8/5}x_2^{1/5}, \qquad \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2 \partial x_1} \;=\; \frac{4}{5}x_1^{-3/5}x_2^{-4/5} \tag{54d}$$

$$\frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1 \partial x_2} \;=\; \frac{4}{5}x_1^{-3/5}x_2^{-4/5}, \qquad \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2 \partial x_2} \;=\; -\frac{8}{5}x_1^{2/5}x_2^{-9/5} \tag{54e}$$

$$\frac{\partial g(x_1, x_2)}{\partial x_1} = 8, \qquad \frac{\partial g(x_1, x_2)}{\partial x_2} \;=\; 2 \tag{54f}$$

This gives for the bordered Hessian matrix

$$\begin{bmatrix} -\frac{12}{5}x_1^{-8/5}x_2^{1/5} & \frac{4}{5}x_1^{-3/5}x_2^{-4/5} & 8 \\ \frac{4}{5}x_1^{-3/5}x_2^{-4/5} & -\frac{8}{5}x_1^{2/5}x_2^{-9/5} & 2 \\ 8 & 2 & 0 \end{bmatrix} \tag{55}$$

At the root x = (8,16), this gives

$$\begin{bmatrix} -\frac{12}{5}(8)^{-8/5}(16)^{1/5} & \frac{4}{5}(8)^{-3/5}(16)^{-4/5} & 8 \\ \frac{4}{5}(8)^{-3/5}(16)^{-4/5} & -\frac{8}{5}(8)^{2/5}(16)^{-9/5} & 2 \\ 8 & 2 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{3}{20} & \frac{1}{40} & 8 \\ \frac{1}{40} & -\frac{1}{40} & 2 \\ 8 & 2 & 0 \end{bmatrix} \tag{56}$$

The determinant is

$$\left(-\frac{3}{20}\right)\left(-\frac{1}{40}\right)(0) + \left(\frac{1}{40}\right)(2)(8) + (8)\left(\frac{1}{40}\right)(2) - \left((8)\left(-\frac{1}{40}\right)(8) + \left(-\frac{3}{20}\right)(2)(2) + \left(\frac{1}{40}\right)\left(\frac{1}{40}\right)(0)\right)$$

$$= 0 + \frac{2}{5} + \frac{2}{5} - \left(-\frac{8}{5} + -\frac{3}{5} + 0\right) = \frac{15}{5}$$

$$= 3$$

Given that this is positive, the values $x_1 = 8$ and $x_2 = 16$ maximize utility.