ECONOMIC OPTIMALITY

1. Formal statement of the decision problem

1.1. Statement of the problem.

$$\begin{aligned} &\max_x \; h(x,a) \\ &\text{such that } x \in G(a) \end{aligned} \tag{1}$$

This says that the problem is to maximize the function h which depends on x and a by choosing the levels of x. The parameter a is used to represent all aspects of the problem not under the control of the decision maker in this particular problem. The level of x chosen is a member of a nonempty and compact set G. If the constraint set can represented by a system of equations, standard Lagrangian techniques may be used. If the set is represented by a system of inequalities, then nonlinear programming and Kuhn-Tucker conditions could be used. The optimal values of x are written as a function of a or $x^*(a)$. These are the reduced form or economic expressions we regularly use such as supplies, demands and investment levels.

1.2. Weierstrass Maximum Theorem.

Theorem 1 (Weierstrass)**.** *Let f: $X \to R$ be a continuous real-valued function on a non-empty compact metric space X. Let M = $\sup_{x \in X} f(x)$ and m = $\inf_{x \in X} f(x)$. Then there is a point x^M and a point x^m such that $f(x^M) = M$ and $f(x^m) = m$ (Debreu* [2, p. 16]*).*

For our purposes we can assume that the metric space X is R^n with the usual metric $||x|| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$. The theorem implies that a continuous function on a nonempty closed bounded set achieves a maximum and a minimum in the set.

1.3. The indirect objective or optimal value function. If the optimal values of the decision variables are substituted in the objective function h(·), the indirect objective or optimal value function is obtained. Specifically

$$\begin{aligned} V(a) &= \max_x \; h(x,a) \text{ such that } x \in G(a) \\ &= (h(x^*(a), a) \end{aligned} \tag{2}$$

The indirect objective function gives the optimized value of the objective function as a function of the parameters not controlled by the agent. The values of the decision variables are set at their optimized levels.

For example, for the consumer, if we substitute the optimal consumption choices as functions of prices and income in the utility function, we get the indirect utility function $\psi(p_1, p_2, \ldots, p_n, m)$. For a firm, if we substitute the cost minimizing input quantities as functions of the input prices and the level of output in the expression for cost, we obtain the cost function $C(y, w_1, w_2, \ldots w_n)$. If, for a firm, we substitute the optimal levels of inputs as functions of output and input prices in the expression for profit, we obtain the profit function $\pi(p, w_1, w_2, \ldots w_n)$.

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1.4. **Theorem of the Maximum.** Important continuity results about optimal value functions are given by the theorem of the maximum.

Theorem 2 (Theorem of the Maximum). *Let h(x,a) be a continuous function with a compact range and suppose that the constraint set G(a)is a non-empty, compact-valued, continuous correspondence of a. Then*

(1) *The function V(a) =* $\max_x \{h(x,a) : x \in G(a)\}$ *is continuous*
(2) *The correspondence m(a) =* $\{x \in G(a) : h(x,a) = V(a\}$ *is nonempty, compact valued and upper semi-continuous.*

Proof*: See Berge* [1, p. 116].

m(a) consists of those values of x in the constraint set that yield the optimal value V(a) for a given a.

1.5. **The envelope theorem.** Consider the optimal value or indirect objective function. Assume that it is sufficiently smooth so that differentiation is possible. Then the envelope theorem says the following

$$\frac{\partial V(a)}{\partial a} = \frac{\partial h(x^*(a),a)}{\partial x}\frac{\partial x^*(a)}{\partial a} + \frac{\partial h(x^*(a),a)}{\partial a} \tag{3}$$

if a is a scalar or

$$\frac{\partial V(a)}{\partial a_i} = \frac{\partial h(x^*(a),a)}{\partial x}\frac{\partial x^*(a)}{\partial a_i} + \frac{\partial h(x^*(a),a)}{\partial a_i} \tag{4}$$

if a is a vector. This is basically the result of totally differentiating the indirect objective function with respect to a parameter of the problem. However, the derivative of h with respect to x is usually part of the first order conditions in the direct optimization problem. Thus the derivative of the indirect objective function can usually be simplified using those conditions.

1.5.1. *The case of no constraint.* Consider equation 1 with no constraint. The first order condition is

$$\frac{\partial h(x(a),a)}{\partial x} = 0 \tag{5}$$

which means that 3 simplifies to

$$\frac{\partial V(a)}{\partial a} = \frac{\partial h(x^*(a),a)}{\partial a} \tag{6}$$

1.5.2. *The case of a single equality constraint.* Consider equation 1 with a single equality constraint as follows

$$\max_x h(x,a) \quad \text{such that g(x, a) = 0} \tag{7}$$

Write the Lagrangian for the problem as

$$\mathcal{L}(x,\lambda;a) = h(x,a) - \lambda\, g(x,a) \tag{8}$$

The first order conditions are

$$\mathcal{L}_x = \frac{\partial h(x,a)}{\partial x} - \lambda\frac{\partial g(x,a)}{\partial x} = 0 \tag{9a}$$

$$\mathcal{L}_\lambda = -g(x,a) = 0 \tag{9b}$$

Now consider the indirect objective function for problem 7

$$\begin{aligned} \text{V(a)} &= \mathcal{L}(x^*(a), \lambda^*(a); a) \;=\; h(x^*(a), a) \;-\; \lambda^*(a)\, g(x^*(a), a) \\ &= h(x^*(a), a) \end{aligned} \tag{10}$$

The second equality follows since g(x*(a),a) is identically zero.

The envelope theorem can be derived directly using the first order conditions and the derivatives of the Lagrangian or more indirectly using derivatives of the constraint. Consider first the derivative of 10 using the definition of the optimal $\mathcal{L}$.

$$\begin{aligned} \frac{\partial V(a)}{\partial a_i} &= \frac{\partial \mathcal{L}(x^*(a), \lambda^*(a);\, a)}{\partial x}\,\frac{\partial x^*(a)}{\partial a_i} + \frac{\partial \mathcal{L}(x^*(a),\, \lambda^*(a);\, a)}{\partial \lambda}\,\frac{\partial \lambda^*(a)}{\partial a_i} + \frac{\partial \mathcal{L}(x^*(a), \lambda^*(a);\, a)}{\partial a_i} \\ &= \frac{\partial \mathcal{L}(x^*(a), \lambda^*(a);\, a)}{\partial a_i} \\ &= \frac{\partial h(x^*(a), a)}{\partial a_i} - \lambda^*(a)\frac{\partial g(x^*(a), a)}{\partial a_i} \end{aligned} \tag{11}$$

This can also be obtained by actually carrying out the differentiation implied by equation 10.

$$\frac{\partial V(a)}{\partial a_i} = \frac{\partial h(x^*(a), a)}{\partial x}\,\frac{\partial x^*(a)}{\partial a_i} + \frac{\partial h(x^*(a), a)}{\partial a_i} \tag{12}$$

The question is whether this expression can be simplified due to the fact the vector x must satisfy the first order conditions in 9. First substitute 9a into 12 to obtain

$$\frac{\partial V(a)}{\partial a_i} = \lambda^*(a)\frac{\partial g(x^*(a), a)}{\partial x}\,\frac{\partial x^*(a)}{\partial a_i} + \frac{\partial h(x^*(a), a)}{\partial a_i} \tag{13}$$

The set of first order conditions in 9b do not appear in 13, but the derivative with respect to a_i does appear. Differentiate this condition (which is an identity) to obtain

$$\begin{aligned} &\frac{\partial g(x^*(a), a)}{\partial x}\,\frac{\partial x^*(a)}{\partial a_i} + \frac{\partial g(x^*(a), a)}{\partial a_i} = 0 \\ &\qquad \Rightarrow \frac{\partial g(x^*(a), a)}{\partial x}\,\frac{\partial x^*(a)}{\partial a_i} = -\frac{\partial g(x^*(a), a)}{\partial a_i} \end{aligned} \tag{14}$$

If this is substituted into 13 the envelope theorem result follows

$$\frac{\partial V(a)}{\partial a_i} = -\lambda^*(a)\frac{\partial g(x^*(a), a)}{\partial a_i} + \frac{\partial h(x^*(a), a)}{\partial a_i} \tag{15}$$

Some specific examples will be given later.

1.6. **Optimality conditions when the constraint set is described by inequalities.**

1.6.1. *Description of the problem.* We can often describe the constraint set G(a) using a series of inequalities. Consider the problem to minimize f(x) subject to $g_i(x) \leq 0$ for i = 1, ... ,I_1, $g_i(x) \geq 0$ for i = I_1 + 1, . . . ,I, and $h_j(x) = 0$ for j = 1, . . . ,J, and $x \in X$, where X is an open set in R^n. The problem would be written as follows:

$$\begin{aligned} &\min_{x} f(x,a) \\ &\text{such that} \\ g_i(x,a) &\leq 0, i = 1, 2, \cdots, I_1 \\ g_i(x,a) &\geq 0, i = I_1 + 1, I_1 + 2, \cdots, I \\ h_j(x,a) &= 0, j = 1, 2, \cdots, J \end{aligned} \tag{16}$$

1.6.2. *Optimality conditions.* The necessary conditions for this problem are known as Karush-Kuhn-Tucker conditions and can be expressed as follows where $\bar{x}$ is a feasible solution:

$$\nabla f(\bar{x}) - \Sigma_{i=1}^{I} u_i \nabla g_i(\bar{x}) - \Sigma_{j=1}^{J} \lambda_j \nabla h_j(\bar{x}) = 0 \tag{17a}$$

$$u_i g_i(\bar{x}) = 0, \quad for\ i = 1, \ldots, I \tag{17b}$$

$$u_i \leq 0, \quad for\ i = 1, \ldots, I_1 \tag{17c}$$

$$u_i \geq 0, \quad for\ i = I_1 + 1, \ldots, I \tag{17d}$$

In a minimization problem set up this way, the constraints expressed as greater than will have positive multipliers.

1.6.3. *Problems where the constraints are non-negativity constraints on the variables.* Consider problems of the type: min f(x) subject to $g_i(x) \geq 0$ for i = 1, . . . ,I, $h_j(x) = 0$ for j = 1, . . . ,J, and $x \geq 0$. Such problems with nonnegativity restrictions on the variables frequently arise in practice in situations where prices or quantities are positive. Clearly, the KKT conditions discussed earlier would apply as usual. However, it is sometimes convenient to eliminate the Lagrangian multipliers associated with $x \geq 0$. The conditions are then

$$\begin{aligned} \nabla f(\bar{x}) - \Sigma_{i=1}^{I} u_i \nabla g_i(\bar{x}) - \Sigma_{j=1}^{J} \lambda_j \nabla h_j(\bar{x}) &\geq 0 \\ \left[\nabla f(\bar{x}) - \Sigma_{i=1}^{I} u_i \nabla g_i(\bar{x}) - \Sigma_{j=1}^{J} \lambda_j \nabla h_j(\bar{x})\right]' x &= 0 \\ u_i\, g_i(\bar{x}) &= 0, \quad for\ i = 1, \ldots, I \\ u_i &\geq 0, \quad for\ i = 1, \ldots, I \end{aligned} \tag{18}$$

1.6.4. *Maximization problems.* Finally, consider the problem to *maximize* f(x) subject to $g_i(x) \leq 0$ for i = 1, . . . ,I_1, $g_i(x) \geq 0$ for i = I_1 + 1, . . . ,I, $h_j(x) = 0$ for j = 1,. . . ,J, and $x \in X$, where X is an open set in R^n. The necessary conditions for optimality can be written as follows:

$$\nabla f(\bar{x}) - \Sigma_{i=1}^{I} u_i \nabla g_i(\bar{x}) - \Sigma_{j=1}^{J} \lambda_j \nabla h_j(\bar{x}) = 0 \tag{19a}$$

$$u_i\, g_i(\bar{x}) = 0, \quad for\ i = 1, \ldots, I \tag{19b}$$

$$u_i \geq 0, \quad for\ i = 1, \ldots, I_1 \tag{19c}$$

$$u_i \leq 0, \quad for\ i = I_1 + 1, \ldots, I \tag{19d}$$

In a maximization problem set up in this fashion, the multipliers associated with less than constraints will have a positive sign.

2. Example problems

2.1. Profit maximization in the general case.

a: Technology Consider the technology set or graph represented by

$$T(y,\ x)\ =\ \{(y,\ x):\ \text{y is producible using x}\} \tag{20}$$

b: Economic environment parameters
The uncontrolled parameters are the prices of inputs and outputs. These are considered to be vectors in R^n_+ and R^m_+ Denote these by w and p.

c: Decision (control) variables The decision variables that are controlled by the agent are x and y.

d: Objective function

$$\begin{aligned} &\max_{x,y}\ py\ -\ wx \\ &\quad s.t.\ [x, y]\ \epsilon\ T \end{aligned} \tag{21}$$

e: Indirect objective (optimal value) function
The optimal decision variables are given by x*(p,w) and y*(p,w). The indirect objective function is the profit function and is given by

$$\begin{aligned} \pi\ &=\ \max_{x,y}\ py\ -\ wx \\ &s.t.\ [x, y]\ \epsilon\ T \\ &= py^*(p, w)\ -\ wx^*(p, w) \end{aligned} \tag{22}$$

2.2. Profit maximization with a single output.

a: Technology Consider the production function represented by
y = f(x), x ϵ R^n_+ and y ϵ R^m_+ .

b: Economic environment parameters
The uncontrolled parameters are the prices of inputs and the price of output. The input prices are considered to be a vector in R^n_{++} while p is in R_{++}.Denote these by w and p.

c: Decision (control) variables
The decision variable that is controlled by the agent is x.

d: Objective function

$$\max_{x}\ pf(x)\ -\ wx \tag{23}$$

e: First order conditions

$$p\ \frac{\partial f(x)}{\partial x}\ -\ w\ =\ 0 \tag{24}$$

f: Indirect objective (optimal value) function
The optimal decision variables are given by x*(p,w) and y = f(x*(p,w)). The indirect objective function is the profit function and is given by

$$\begin{aligned} \pi\ &=\ \max_{x}\ pf(x)\ -\ wx \\ &= p\,f\,(x^*(p, w))\ -\ wx^*(p, w) \end{aligned} \tag{25}$$

g: Envelope theorem

We can differentiate the optimal value function with respect to p and w and use the first order conditions to simplify.

$$\begin{aligned}
\frac{\partial \pi(p,w)}{\partial p} &= f\,(x^*(p,w)) \;+\; p\,\frac{\partial f\;(x^*(p,w))}{\partial x}\,\frac{\partial x^*(p,w)}{\partial p} \;-w\,\frac{\partial x^*(p,w)}{\partial p} \\
&= f\;(x^*(p,w)) \;+\; \left(p\,\frac{\partial f\;(x^*(p,w))}{\partial x} \;-w \right) \frac{\partial x^*(p,w)}{\partial p} \\
&= f\;(x^*(p,w)) \;=\; y^*(p,w) \\
\frac{\partial \pi(p,w)}{\partial w_i} &= p\,\frac{\partial f\;(x^*(p,w))}{\partial x}\,\frac{\partial x^*(p,w)}{\partial w_i} \;-w\,\frac{\partial x^*(p,w)}{\partial w_i} \;-\; x_i^*(p,w) \\
&= \left(p\,\frac{\partial f\;(x^*(p,w))}{\partial x} \;-w \right) \frac{\partial x^*(p,w)}{\partial w_i} \;-\; x_i^*(p,w) \\
&= -\; x_i^*(p,w)
\end{aligned} \tag{26}$$

2.3. **Profit maximization with a single output and fixed inputs.**

a: Technology

Consider the production function represented by y = f(x,z), x $\epsilon\ R^n_+$, z $\epsilon\ R^k_+$, and y $\epsilon\ R^m_+$ where the inputs z are fixed.

b: Economic environment parameters

The uncontrolled parameters are the prices of inputs, the price of output, and the fixed inputs. The input prices for the variable inputs are considered to be a vector in R^n_{++} , z an element of R^k_+ , the prices of z, denoted by r, an element of R^k_{++} , while output price p is in R^1_{++} . Denote the prices of x by w as before.

c: Decision (control) variables

The decision variable that is controlled by the agent is the vector x.

d: Objective function

$$\max_x \; pf(x,z) \;-\; wx \;-\; rz \tag{27}$$

e: First order conditions

$$p\,\frac{\partial f(x,z)}{\partial x} \;-\; w \;=\; 0 \tag{28}$$

f: Indirect objective (optimal value) function

The optimal decision variables are given by x*(p,w,z) and y = f(x*(p,w,z),z). The indirect objective function is the profit function and is given by

$$\begin{aligned}
\pi(p,w,z) \;&=\; \max_x \; pf(x,z) \;-\; wx \;-\; rz \\
&= p\,f\,(x^*(p,w,z),z) \;-\; wx^*(p,w,z) \;-\; rz
\end{aligned} \tag{29}$$

g: Envelope theorem

We can differentiate the optimal value function with respect to p and w and use the first order conditions to simplify. The answers will be basically the same as in the previous case because the term rz drops out of the derivatives. We can also consider the effect of z on the optimal value function.

$$\begin{aligned}\frac{\partial \pi(p,w,z)}{\partial z_i} &= p\,\frac{\partial f\ (x^*(p,w,z)\,,\,z)}{\partial x}\,\frac{\partial x^*(p,w\,,\,z)}{\partial z_i} + p\,\frac{\partial f\ (x^*(p,w\,,\,z),z)}{\partial z_i} - w\,\frac{\partial x^*(p,w,z)}{\partial z_i} - r_i \\ &= \left(p\,\frac{\partial f\ (x^*(p,w,z)\,,\,z)}{\partial x} - w \right) \frac{\partial x^*(p,w\,,\,z)}{\partial z_i} + p\,\frac{\partial f\ (x^*(p,w,z)\,,\,z)}{\partial z_i} - r_i \\ &= p\,\frac{\partial f\ (x^*(p,w,z)\,,\,z)}{\partial z_i} - r_i\end{aligned} \tag{30}$$

This just says that as the quantity of the ith fixed input increases, profits change by the difference between its marginal value product and its cost. This could be regarded as a net shadow price for this input.

2.4. Cost minimization in the general case.

a: Technology

Consider the technology set or graph represented by

$$\text{T(y, x)} = \{\text{(y, x)} : \text{ y is producible using x}\} \tag{31}$$

Since y will be considered fixed for this problem, consider the representation in terms of the input correspondence V and denote the technological possibilities by V(y).

b: Economic environment parameters

The uncontrolled parameters are the prices of inputs and the fixed output levels. These are considered to be vectors in R^n_{++} and R^m_{++} . Denote input prices by w.

c: Decision (control) variables

The decision variables that are controlled by the agent are the x's.

d: Objective function

$$\begin{aligned}&\min_x \ wx \\ &s.t.\ x \ \epsilon \ V(y)\end{aligned} \tag{32}$$

e: Indirect objective (optimal value) function

The optimal decision variables are given by x*(y,w). The indirect objective function is the cost function and is given by

$$\begin{aligned}C(y,w) &= \min_x \ wx \\ &\quad s.t.\ x \ \epsilon \ V(y) \\ &= wx^*(y,w)\end{aligned} \tag{33}$$

2.5. Cost minimization with a single output.

a: Technology

Consider the production function represented by y = f(x), x ϵ R^n_+ and y ϵ R^1_+ .

b: Economic environment parameters

The uncontrolled parameters are the prices of inputs and the level of the output. The input prices are considered to be a vector in R^n_{++} while y is in R^m_+ . Input prices are denoted w.

c: Decision (control) variables

The decision variable that is controlled by the agent is x.

d: Objective function

$$\begin{aligned} &\min_x wx \\ &s.t.\, y = f(x) \end{aligned} \tag{34}$$

e: First order conditions
Set up a Lagrangian as follows

$$\mathcal{L} = wx - \lambda(f(x) - y)$$

The first order conditions are given by

$$\begin{aligned} w - \lambda \frac{\partial f(x)}{\partial x} &= 0 \\ -f(x) + y &= 0 \end{aligned} \tag{35}$$

f: Indirect objective (optimal value) function
The optimal decision variables are given by x*(y,w). The indirect objective function is the cost function and is given by

$$\begin{aligned} C(y,w) &= \min_x \; wx \\ &s.t. \; y = f(x) \\ &= wx^*(y,w) \end{aligned} \tag{36}$$

g: Envelope theorem
We can differentiate the optimal value function with respect to y and w and use the first order conditions to simplify

$$\frac{\partial C(y,w)}{\partial w_i} = x_i^*(y,w) + w \frac{\partial x^*(y,w)}{\partial w_i} \tag{37a}$$

$$= x_i^*(y,w) + \lambda \frac{\partial f(x^*(y,w))}{\partial x} \frac{\partial x^*(y,w)}{\partial w_i} \tag{37b}$$

Now differentiate equation 37b with respect to w_i

$$\begin{aligned} -f(x^*(y,w)) + y &= 0 \\ \Rightarrow -\frac{\partial f(x^*(y,w))}{\partial x} \frac{\partial x^*(y,w)}{\partial w_i} &= 0 \\ \Rightarrow \frac{\partial C(y,w)}{\partial w_i} &= x_i^*(y,w) \end{aligned} \tag{38}$$

Similarly for the derivative with respect to y

$$\begin{aligned} \frac{\partial C(y,w)}{\partial y} &= w \frac{\partial x^*(y,w)}{\partial y} \\ &= \lambda \frac{\partial f(x^*(y,w))}{\partial x} \frac{\partial x^*(y,w)}{\partial y} \end{aligned} \tag{39}$$

Now differentiate equation 37b with respect to y.

$$\begin{aligned} -f(x^*(y,w)) \;+\; y \;&= 0 \\ \Rightarrow\; -\frac{\partial f(x^*(y,w))}{\partial x}\frac{\partial x^*(y,w)}{\partial y} \;+\; 1 \;&= 0 \\ \Rightarrow\; \frac{\partial C(y,w)}{\partial y} \;&= \lambda \end{aligned} \tag{40}$$

2.6. **Utility maximization.**

a: Preferences

Preferences are represented by
u = v(x), x ϵ R^n_+ and u ϵ R^1_+ .

b: Economic environment parameters

The uncontrolled parameters are the prices of goods and the level of the income. The prices are considered to be a vector in R^n_{++} while u is in R^m_+ . Prices are denoted p.

c: Decision (control) variables

The decision variable that is controlled by the agent is x.

d: Objective function

$$\begin{aligned} &\max_x v\,(\,x\,) \\ &s.t.\; \Sigma_{j=1}^n\, p_j\, x_j \;=\; m \end{aligned} \tag{41}$$

e: First order conditions

Set up a Lagrangian as follows

$$\mathcal{L} \;=\; v\,(\,x\,) \;-\; \lambda\,(\,p\,x \;-\; m)$$

The first order conditions are given by

$$\frac{\partial\, v\,(x)}{\partial x} \;-\; \lambda\, p \;= 0 \tag{42a}$$

$$-p\,x \;+\; m \;= 0 \tag{42b}$$

f: Indirect objective (optimal value) function

The optimal decision variables are given by x*(p,m). The indirect objective function is the indirect utility function and is given by

$$\begin{aligned} \psi\,(p,\; m) \;=\; &\max_x v\,(\,x\,) \\ &s.t.\; \Sigma_{j=1}^n\, p_j\, x_j \;=\; m \\ &= v\;(x^*\,(p,\; m\,)\;) \end{aligned} \tag{43}$$

g: Envelope theorem

We can differentiate the optimal value function with respect to p and m and use the first order conditions to simplify

$$\frac{\partial\,\psi(\,p,\,m)}{\partial\, p_i} \;= \Sigma_{j=1}^n\, \frac{\partial\, v\,(x^*\,(p,\; m)}{\partial\, x_j}\; \frac{\partial x_j^*\,(p\,,\; m)}{\partial\, p_i} \tag{44a}$$

$$\frac{\partial\,\psi(\,p,\,m)}{\partial\, m} \;= \Sigma_{j=1}^n\, \frac{\partial\, v\,(x^*\,(p,\; m)}{\partial\, x_j}\; \frac{\partial x_j^*\,(p\,,\; m)}{\partial\, m} \tag{44b}$$

Now substitute from equation 42a for $\frac{\partial v(x)}{\partial x_j} = \lambda p_j$ so that we obtain

$$\frac{\partial \psi(p, m)}{\partial p_i} = \lambda \Sigma_{j=1}^{n} p_j \frac{\partial x_j^*(p, m)}{\partial p_i} \tag{45a}$$

$$\frac{\partial \psi(p, m)}{\partial m} = \lambda \Sigma_{j=1}^{n} p_j \frac{\partial x_j^*(p, m)}{\partial m} \tag{45b}$$

Now differentiate equation 42b with respect to m.

$$\begin{aligned} -\Sigma_{k=1}^{n} p_{k\, x_k^*}(p, m) + m &= 0 \\ -\Sigma_{k=1}^{n} p_k \frac{\partial x_k^*}{\partial m} + 1 &= 0 \\ \Rightarrow \Sigma_{k=1}^{n} p_k \frac{\partial x_k^*}{\partial m} &= 1 \end{aligned} \tag{46}$$

Now differentiate equation 42b with respect to p_i.

$$\begin{aligned} -\Sigma_{k=1}^{n} p_{k\, x_k^*}(p, m) + m &= 0 \\ -\Sigma_{k=1}^{n} p_k \frac{\partial x_k^*}{\partial p_i} - x_i^*(p, m) &= 0 \\ \Rightarrow x_i^*(p, m) &= -\Sigma_{k=1}^{n} p_k \frac{\partial x_k^*}{\partial p_i} \end{aligned} \tag{47}$$

Now simplify 45 using 46 and 47

$$\begin{aligned} \frac{\partial \psi(p, m)}{\partial p_i} &= \lambda \Sigma_{j=1}^{n} p_j \frac{\partial x_j^*(p, m)}{\partial p_i} \\ &= \lambda (-x_i^*(p, m)) \\ \frac{\partial \psi(p, m)}{\partial m} &= \lambda \Sigma_{j=1}^{n} p_j \frac{\partial x_j^*(p, m)}{\partial m} \\ &= \lambda (1) \end{aligned} \tag{48}$$

If we take the ratio of the two equations in 48 we obtain

$$\frac{\frac{\partial \psi(p, m)}{\partial p_i}}{\frac{\partial \psi(p, m)}{\partial m}} = -x_i^*(p, m)) \tag{49}$$

This is called Roy's identity.

2.7. **Revenue maximization in the general case.**

a: Technology

Consider the technology set or graph represented by

$$\mathrm{T(y, x)} = \{(\mathrm{y, x}) : \text{ y is producible using x}\} \tag{50}$$

Since x will be considered fixed for this problem, consider the representation in terms of the output correspondence P and denote the technological possibilities by P(x).

b: Economic environment parameters

The uncontrolled parameters are the prices of outputs and the fixed input levels. These are considered to be vectors in R^m_{++} and R^n_+ . Denote output prices by p.

c: Decision (control) variables

The decision variables that are controlled by the agent are the y's.

d: Objective function

$$\max_y \; py$$
$$s.t. \; y \,\epsilon\, P(x) \tag{51}$$

e: Indirect objective (optimal value) function

The optimal decision variables are given by y*(x,p). The indirect objective function is the revenue function and is given by

$$\begin{aligned} R(x,p) &= \max_y \; py \\ &\quad s.t. \; y \,\epsilon\, P(x) \\ &= py^*(x,p) \end{aligned} \tag{52}$$

2.8. **Revenue maximization with cost constrained production.**

a: Technology

Consider the following technology which represents all vectors y which can be produced at cost less than or equal to C where w is the vector of input prices.

$$\mathrm{IP(w/C)} \; : \; \mathrm{R^n_+} \rightarrow 2^{\mathrm{R^m_+}} \; = \; \{\mathrm{y} : \; \mathrm{C(y,w)} \; \leq \; \mathrm{C}\} \; = \; \{\mathrm{y} : \; \mathrm{C(y,w/C)} \; \leq \; 1\}$$

This can also be represented as

IP(w/C) ={yϵP(x): x ϵ R^n_+ , wx $\leq$ C}

This correspondence is called the cost indirect output correspondence. In the case of a single output, its maximum element would define the cost indirect production function and would give the maximum level of output attainable for a given expenditure.

b: Economic environment parameters

The uncontrolled parameters are the normalized input prices and output prices. These are considered to be vectors in R^n_{++} and R^m_{++} . Denote output prices by p.

c: Decision (control) variables

The decision variables that are controlled by the agent are the y's.

d: Objective function

$$\begin{aligned} &\sup_y \; [py : y \,\epsilon\, IP(w/C)] \\ = \; &\sup_y \; [py : \; C(y,w) \; \leq \; C] \end{aligned} \tag{53}$$

e: Indirect objective (optimal value) function

The optimal decision variables are given by y*(w/C,p). The indirect objective function is the cost indirect revenue function, which gives the maximum revenue attainable for a given level of expenditure. It is given by

$$\begin{aligned} IR(w/C, p) &= \sup_y \, [py : \; y \, \epsilon \, IP(w/C) \\ &= py^*(w/C, \;p) \end{aligned} \tag{54}$$

3. Optimization and observed economic data

3.1. **Assumption of Optimality.** Economists usually assume that economic data is generated by agents solving optimization problems similar to those discussed in this lecture. This then implies that the observed data should satisfy properties imposed by the technological assumptions, institutional and other economic constraints and the optimization hypothesis. Tests of various hypotheses can be made using these relationships.

3.2. **Assumption of endogeneity.** Given that observed economic data is the result of firm level optimization problems, usual classifications of endogenous and exogenous variables may not be appropriate. For example, observed levels of inputs and outputs may not be appropriate to estimate a production function using ordinary least squares since they satisfy a set of first order conditions related to the firm level decision problem.

3.3. **Duality.** There is a duality relationship between underlying direct technological correspondences and derived optimal decision functions. The form of this duality will differ in various cases but usually is stated in the form of conjugate pairs of functions that can be derived from each other such as the cost and distance functions, or convex sets and intersections of supporting half-spaces such as the input correspondence and the half-spaces defined by the cost function. In cases of duality, either of the two formulations is an equivalent representation of the technology.

3.4. **Relevance of envelope theorem.** In many problems, the derivative of the indirect objective function with respect to parameters gives optimal levels of decision variables as functions of those parameters.

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